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$N = 4$ supergravity lagrangian for type-IIB on $T^6/\mathbb{Z}_2$ orientifold in presence of fluxes and $D3$-branes

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Abstract: We derive the lagrangian and the transformation laws of $N = 4$ gauged supergravity coupled to matter multiplets whose σ-model of the scalars is $\mathrm{SU}(1,1)/\mathrm{U}(1) \otimes \mathrm{SO}(6,6+n)/\mathrm{SO}(6) \otimes \mathrm{SO}(6+n)$ and which corresponds to the effective lagrangian of the type-IIB string compactified on the $T^6/\mathbb{Z}_2$ orientifold with fluxes turned on and in presence of n $D3$-branes. The gauge group is $T^{12} \otimes G$ where G is the gauge group on the brane and T^{12} is the gauge group on the bulk corresponding to the gauged translations of the R-R scalars coming from the R-R four-form.
The $N = 4$ bulk sector of this theory can be obtained as a truncation of the Scherk-Schwarz spontaneously broken $N = 8$ supergravity. Consequently the full bulk spectrum satisfies quadratic and quartic mass sum rules, identical to those encountered in Scherk-Schwarz reduction gauging a flat group.
This theory gives rise to a no scale supergravity extended with partial super-Higgs mechanism.

Keywords: Superstring Vacua, Supersymmetry Breaking, Supergravity Models.

Contents

1. Introduction

In recent time, compactification of higher dimensional theories in presence of p-form fluxes [1]–[23] has given origin to new four-dimensional vacua with spontaneously broken supersymmetry and with vanishing vacuum energy. These models realize, at least at the classical level, the no-scale structure [24]–[26] of extended supergravities in an M or String theory setting [27]–[34].

No-scale supergravities also arise from Scherk-Schwarz generalized dimensional reduction [35]–[37], where a flat group is gauged.

From a pure four-dimensional point of view all these models can be viewed as particular cases of gauged-extended supergravities (for recent reviews see [38]–[40]). The gauge couplings correspond to fluxes turned on.[1] This is so because for $N > 1$ supersymmetry a scalar potential is necessarily due to the presence of gauge symmetries. It has been shown that a common feature of all no-scale structures is that the complete gauge group of the theory contains a sector where "axionic" symmetries are gauged [41]–[47].

The Higgs effect in this sector is then tightly connected to the super-Higgs mechanism [48, 49]. The complete gauge group is usually larger than this sector and the additional gauge bosons are frequently associated to central charges of the supersymmetry algebra.

For instance, in $N = 8$ spontaneously broken supergravity à la Scherk-Schwarz, the translational part T_Λ is 27-dimensional and the extra sector T_0 is one-dimensional, then completing a 28-dimensional flat group [35, 44]

$$[T_\Lambda, T_0] = f^{\Delta}_{\Lambda 0} T_\Delta\,; \qquad [T_\Lambda, T_\Sigma] = 0\,; \qquad \Lambda, \Sigma = 1 \ldots 27\,. \tag{1.1}$$

This algebra is a 28-dimensional subalgebra of $\mathfrak{e}_{7,7}$. In the case of the IIB orientifold $T^6/\mathbb{Z}_2$, the translational part T_Λ is 12-dimensional, while the extra sector T_i are the Yang-Mills generators on the brane [8, 9, 50]

$$[T_\Lambda, T_i] = 0\,; \quad [T_\Lambda, T_\Sigma] = 0\,; \quad [T_i, T_j] = c_{ij}{}^k T_k\,; \quad \Lambda\Sigma = 1 \ldots 12;\, i, j, k = 1 \ldots \dim G\,. \tag{1.2}$$

What is common to these groups is that they must have a symplectic action on the vector field strengths and their dual [51]. This implies that they must be embedded in $\mathrm{Sp}(2n, \mathbb{R})$, where $n = 12 + \dim G$ in the orientifold case.

The particular choice of the embedding determines the structure of the gauged supergravity. In the case of the type-IIB supergravity in presence of $D3$-branes, the strong requirement is that the original $\mathrm{SL}(2, \mathbb{R})$ symmetry acts linearly on the twelve bulk vectors $(B_{\mu\Lambda}, C_{\mu\Lambda})$, $\Lambda = 1 \ldots 6$, but acts as an electric magnetic duality on the vectors A^i_μ, $i = 1 \ldots n$ living on the $D3$-branes.[2]

Mathematically this corresponds to a very particular embedding of $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SO}(6, 6+n)$ into $\mathrm{Sp}(24 + 2n, \mathbb{R})$.

The relevant decomposition is

$$\mathfrak{so}(6, 6+n) = \mathfrak{sl}(6, \mathbb{R})^0 + \mathfrak{so}(1,1)^0 + \mathfrak{so}(n)^0 + (\mathbf{15'}, \mathbf{1})^{+2} + (\mathbf{15}, \mathbf{1})^{-2} + (\mathbf{6'}, \mathbf{n})^{+1} + (\mathbf{6}, \mathbf{n})^{-1} \tag{1.3}$$

where $\mathfrak{so}(n) \supset Adj\, G_{(\dim n)}$ (note that if $G = \mathrm{U}(N)$, then $n = N^2$). The symplectic embedding of the $12 + n$ vectors such that $\mathrm{SL}(2, \mathbb{R})$ is diagonal on 12 vectors and off diagonal in the remaining Yang-Mills vectors on the branes, is performed in section 3.

Interestingly, the full bulk sector of the $T^6/\mathbb{Z}_2$ type IIB orientifold can be related to a $N = 4$ truncation of the $N = 8$ spontaneously broken supergravity à la Scherk-Schwarz [35, 36]. This will be proven in detail in section 8.

[1] Note that in String and M theories the fluxes satisfy some quantization conditions [1]–[23]

[2] $(B_{\mu I}, C_{\mu I})$ are the $\mathrm{SL}(2, \mathbb{R})$ doublet N-S and R-R two-forms with one leg on space-time and one leg on the torus

The U(4) R-symmetry of the type-IIB theory is identified with the U(4) $\subset$ USp(8) of the $N=8$ theory, while SL$(2,\mathbb{R}) \times$ GL(6) is related to the subgroup of $\mathrm{E}_{6(6)} \times \mathrm{SO}(1,1) \subset \mathrm{E}_{7(7)}$. The $N=4$ truncation is obtained by deleting the left-handed gravitino in the $\overline{\mathbf{4}}^{\,-\frac{1}{2}}$ and keeping the $\mathbf{4}^{+\frac{1}{2}}$ in the decomposition of the $\mathbf{8}$ of USp(8) into U(4) irreducible representations: $\mathbf{8} \longrightarrow \overline{\mathbf{4}}^{\,-\frac{1}{2}} + \mathbf{4}^{+\frac{1}{2}}$.

The $N=8$ gravitino mass matrix (the $\mathbf{36}$ of USp(8)) decomposes as follows

$$\mathbf{36} \longrightarrow \mathbf{1}^0 + \mathbf{15}^0 + \mathbf{10}^{+1} + \overline{\mathbf{10}}^{\,-1} \tag{1.4}$$

and the representation $\mathbf{10}^{+1}$ corresponds to the $N=4$ gravitino mass matrix of the orientifold theory [50, 57].

The vacuum condition of the $N=8$ Scherk-Schwarz model corresponds to the vanishing of a certain representation $\mathbf{42}$ of USp(8) [37, 44]. Its $N=4$ decomposition is

$$\mathbf{42} \longrightarrow \mathbf{20}^0 + \mathbf{1}^{+2} + \mathbf{1}^{-2} + \overline{\mathbf{10}}^{\,+1} + \mathbf{10}^{-1} \tag{1.5}$$

and the vacuum condition of the $N=4$ orientifold theory corresponds to setting to zero [50, 57] the representation $\mathbf{10}^{-1}$ (the other representations being deleted in the truncation).

This theory has a six-dimensional moduli space ($6+6N$, N being the dimensional of the Cartan subalgebra of G, if the $D3$-brane coordinates are added) which is locally three copies of SU(1,1)/U(1) [45, 52, 50]. The spectrum depends on the overall scale $\gamma = (R_1 R_2 R_3)^{-1} = e^{\frac{K}{2}}$, where K is the Kähler potential of the moduli space. In units of this scale, if we call m_i ($i=1,2,3,4$) the four gravitino masses, the overall mass spectrum has a surprisingly simple form, and in fact it coincides with a particular truncation (to half of the states) of the $N=8$ spectrum of Scherk-Schwarz spontaneously broken supergravity [36].

The mass spectrum satisfies the quadratic and quartic relations:

$$\begin{aligned} \sum_J (2J+1)(-1)^{2J} m_J^2 &= 0 \\ \sum_J (2J+1)(-1)^{2J} m_J^4 &= 0\,. \end{aligned} \tag{1.6}$$

These relations imply that the one-loop divergent contribution to the vacuum energy is absent, in the field theory approximation [53, 54]. In the present investigation we complete the analysis performed in reference [57, 50]. In these previous works the part referring to the bulk sector of the theory and the vacua in presence of $D3$-branes degrees of freedom were obtained.

The paper is organized as follows:

- In section 2 we describe the $N=4$ σ-model geometry of the bulk sector coupled to n $D3$-branes.
- In section 3 we give in detail the symplectic embedding which describes the bulk IIB theory coupled to $D3$-brane gauge fields.
- In section 4 the gauging of the $N=4$ theory is given.

- In section 5 the lagrangian (up to four fermions terms) and the supersymmetry transformation laws (up to three fermions terms) are obtained.
- In section 6 the potential and its extrema are discussed.
- In section 7 the mass spectrum is given.
- In section 8 we describe the embedding of our model in the $N = 8$ supergravity and its relation with the Scherk-Schwarz compactification.
- In appendix A we describe the geometric method of the Bianchi identities in superspace in order to find the supersymmetry transformation laws on space-time.
- In appendix B we use the geometric method (rheonomic approach)in order to find a superspace lagrangian which reduces to the space-time lagrangian after suitable projection on the space-time.
- In appendix C we give a more detailed discussion of the freezing of the moduli when we reduce in steps $N = 4 \longrightarrow 3, 2, 1, 0$ using holomorphic coordinates on the T^6 torus.
- In appendix D we give some conventions.

2. The geometry of the scalar sector of the $T^6/\mathbb{Z}_2$ orientifold in presence of $D3$-branes

2.1 The σ-model of the bulk supergravity sector

For the sake of establishing notations, let us first recall the physical content of the $N = 4$ matter coupled supergravity theory.

The gravitational multiplet is

$$\{V^a_\mu;\ \psi_{A\mu};\ \psi^A_\mu;\ A^I_{1\mu};\ \chi^A;\ \chi_A;\ \phi_1;\ \phi_2\} \tag{2.1}$$

where $\psi_{A\mu}$ and ψ^A_μ are chiral and antichiral gravitini, while χ^A and χ_A are chiral and antichiral dilatini; V^a_μ is the vierbein, $A^I_{1\mu}$, $I = 1, \dots 6$ are the graviphotons and the complex scalar fields ϕ_1, ϕ_2 satisfy the constraint $\phi_1\overline{\phi}_1 - \phi_2\overline{\phi}_2 = 1$.

We also introduce $6 + n$ Yang-Mills vector multiplets, from which 6 will be considered as vector multiplets of the bulk, namely

$$\{A^I_{2\mu};\ \lambda^I_A;\ \lambda^{IA};\ s^r\} \tag{2.2}$$

where λ^I_A and λ^{IA} are respectively chiral and antichiral gaugini, $A^I_{2\mu}$ are matter vectors and s^r, $r = 1, \dots 36$ are real scalar fields.

Correspondingly we denote the n vector multiplets, which microscopically live on the $D3$-branes, as

$$\{A^i_\mu;\ \lambda^i_A;\ \lambda^{iA};\ q^I_i\} \tag{2.3}$$

where $i = 1, \dots n$.

It is well known that the scalar manifold of the $N = 4$ supergravity coupled to $6 + n$ vector multiplets is given by the coset space [55, 56]

$$\frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)} \otimes \frac{\mathrm{SO}(6,6+n)}{\mathrm{SO}(6) \times \mathrm{SO}(6+n)} . \tag{2.4}$$

Denoting by $w[\ ]$, the weights of the fields under the U(1) factor of the U(4) R-symmetry, the weights of the chiral spinors are[3]

$$w[\psi_A] = \frac{1}{2}; \qquad w[\chi^A] = \frac{3}{2}; \qquad w[\lambda_{IA}] = -\frac{1}{2}; \qquad w[\lambda_{iA}] = -\frac{1}{2} \tag{2.5}$$

and for the $\mathrm{SU}(1,1)/\mathrm{U}(1)$ scalars we have

$$w[\phi_1] = w[\phi_2] = -1; \qquad w\left[\overline{\phi}_1\right] = w\left[\overline{\phi}_2\right] = 1 . \tag{2.6}$$

Let us now describe the geometry of the coset σ-model.

For the $\mathrm{SU}(1,1)/\mathrm{U}(1)$ factor of the $N = 4$ σ-model we use the following parameterization [57]:

$$S_{\mathrm{SU}(1,1)} = \begin{pmatrix} \phi_1 & \overline{\phi}_2 \\ \phi_2 & \overline{\phi}_1 \end{pmatrix} \qquad (\phi_1\overline{\phi}_1 - \phi_2\overline{\phi}_2 = 1) \tag{2.7}$$

Introducing the 2-vector

$$\begin{gathered} \begin{pmatrix} L^1 \\ L^2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_1 + \phi_2 \\ -i(\phi_1 - \phi_2) \end{pmatrix} \\ w[L^\alpha] = -1; \qquad w[\overline{L}^\alpha] = 1 \end{gathered} \tag{2.8}$$

the identity $\phi_1\overline{\phi}_1 - \phi_2\overline{\phi}_2 = 1$ becomes:

$$L^\alpha \overline{L}^\beta - \overline{L}^\alpha L^\beta = i\epsilon^{\alpha\beta} \tag{2.9}$$

The indices $\alpha = 1, 2$ are lowered by the Ricci tensor $\epsilon_{\alpha\beta}$, namely:

$$L_\alpha \equiv \epsilon_{\alpha\beta} L^\beta . \tag{2.10}$$

A useful parametrization of the $\mathrm{SU}(1,1)/\mathrm{U}(1)$ coset is in terms of the N-S, R-R string dilatons of type-IIB theory [50]

$$\frac{\phi_2}{\phi_1} = \frac{i - S}{i + S} \tag{2.11}$$

with $S = ie^\varphi + C$, from which follows, fixing an arbitrary U(1) phase:

$$S = -\frac{L^2}{L^1} \tag{2.12}$$

$$\phi_1 = -\frac{1}{2}[i(e^\varphi + 1) + C]e^{-\frac{\varphi}{2}} \tag{2.13}$$

$$\phi_2 = \frac{1}{2}[i(e^\varphi - 1) + C]e^{-\frac{\varphi}{2}} \tag{2.14}$$

$$L^1 = -\frac{i}{\sqrt{2}} e^{-\frac{\varphi}{2}} \tag{2.15}$$

$$L^2 = -\frac{1}{\sqrt{2}} \left(e^{\frac{\varphi}{2}} - iCe^{-\frac{\varphi}{2}} \right) . \tag{2.16}$$

Note that the physical complex dilaton S is U(1) independent.

[3]Throughout the paper lower SU(4) indices belong to the fundamental representation, while upper SU(4) indices belong to its complex conjugate

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We will also use the isomorphism $\mathrm{SU}(1,1) \sim \mathrm{SL}(2,\mathbb{R})$ realized with the Cayley matrix $\mathcal{C}$

$$\mathcal{C} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \tag{2.17}$$

$$S_{\mathrm{SL}(2,\mathbb{R})} = \mathcal{C} S_{\mathrm{SU}(1,1)} \mathcal{C}^{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} L^1 + \overline{L}^1 & i(L^1 - \overline{L}^1) \\ L^2 + \overline{L}^2 & i(L^2 - \overline{L}^2) \end{pmatrix} \equiv \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}. \tag{2.18}$$

We note that the 2-vector $(L^\alpha\ \overline{L}^\alpha)$ transform as a vector of $\mathrm{SL}(2,\mathbb{R})$ on the left and $\mathrm{SU}(1,1)$ on the right. Indeed:

$$\widetilde{S} = \mathcal{C} S_{\mathrm{SU}(1,1)} = \begin{pmatrix} L^1 & \overline{L}^1 \\ L^2 & \overline{L}^2 \end{pmatrix}. \tag{2.19}$$

The left-invariant Lie algebra valued 1-form of $\mathrm{SU}(1,1)$is defined by:

$$\theta \equiv S^{-1} dS = \begin{pmatrix} q & \overline{p} \\ p & -q \end{pmatrix} \tag{2.20}$$

where the coset connection 1-form q and the vielbein 1-form p are given by:

$$q = i\epsilon_{\alpha\beta} L^\alpha d\overline{L}^\beta \tag{2.21}$$

$$p = -i\epsilon_{\alpha\beta} L^\alpha dL^\beta\,. \tag{2.22}$$

Note that we have the following relations

$$\nabla L^\alpha \equiv dL^\alpha + qL^\alpha = -\overline{L}^\alpha p \tag{2.23}$$

$$\nabla \overline{L}^\alpha \equiv d\overline{L}^\alpha - q\overline{L}^\alpha = -L^\alpha \overline{p}\,. \tag{2.24}$$

To discuss the geometry of the $\mathrm{SO}(6,6+n)/\mathrm{SO}(6)\times\mathrm{SO}(6+n)$ σ-model, it is convenient to consider first the case $n=0$, that is the case when only six out the $6+n$ vector multiplets are present (no $D3$-branes). This case was studied in reference [57].

In this case the coset reduces to $\frac{\mathrm{SO}(6,6)}{\mathrm{SO}(6)\times\mathrm{SO}(6)}$; with respect to the subgroup $\mathrm{SL}(6,\mathbb{R}) \times \mathrm{SO}(1,1)$ the $\mathrm{SO}(6,6)$ generators decompose as follows:

$$\mathfrak{so}(6,6) = \mathfrak{sl}(6,\mathbb{R})^0 + \mathfrak{so}(1,1)^0 + (\mathbf{15}',\mathbf{1})^{+2} + (\mathbf{15},\mathbf{1})^{-2} \tag{2.25}$$

where the superscripts refer to the $\mathrm{SO}(1,1)$ grading. We work in the basis where the $\mathrm{SO}(6,6)$ invariant metric has the following form

$$\eta = \begin{pmatrix} \mathbf{0}_{6\times 6} & \mathbf{1}_{6\times 6} \\ \mathbf{1}_{6\times 6} & \mathbf{0}_{6\times 6} \end{pmatrix}. \tag{2.26}$$

Thus, the generators in the right hand side of (2.25) are:

$$\begin{aligned} &\mathfrak{sl}(6,\mathbb{R}) : \begin{pmatrix} A & 0 \\ 0 & -A^T \end{pmatrix} \qquad \mathfrak{so}(1,1) : \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \\ &(\mathbf{15}',\mathbf{1})^{+2} : T_{[\Lambda\Sigma]} = \begin{pmatrix} 0 & t_{[\Lambda\Sigma]} \\ 0 & 0 \end{pmatrix} \qquad (\mathbf{15},\mathbf{1})^{-2} : (T_{[\Lambda\Sigma]})^T \end{aligned} \tag{2.27}$$

where we have defined:

$$t_{[\Lambda\Sigma]}{}^{\Gamma\Delta} = \delta^{\Gamma\Delta}_{\Lambda\Sigma}\,; \qquad \Lambda,\,\Sigma = 1,\ldots,6 \tag{2.28}$$

and A are the $\mathrm{SL}(6,\mathbb{R})$ generators. It is useful to split the scalar fields s^r into those which span the $\mathrm{GL}(6,\mathbb{R})/\mathrm{SO}(6)_d$ submanifold and which parametrize the corresponding coset representative $L_{\mathrm{GL}(6,\mathbb{R})}$ from the axions parametrizing the $\mathbf{15}'^{+2}$ translations. We indicate them respectively with $E = E^T \equiv E^I_\Lambda$, $E^{-1} \equiv (E^{-1})^\Lambda_I$ symmetric 6×6 matrices and with $B = -B^T \equiv B^{\Lambda\Sigma}$, $\Lambda,\,\Sigma = 1,\ldots,6$, $I = 1,\ldots,6$. Note that the capital Greek indices refer to global GL(6) while the capital Latin indices refer to local $\mathrm{SO}(6)_{(d)}$ transformations. The coset representatives $L_{\mathrm{GL}(6,\mathbb{R})}$ and the full coset representative L can thus be constructed as follows:

$$\begin{aligned} L = \exp\left(-B^{\Lambda\Sigma}T_{[\Lambda\Sigma]}\right)L_{\mathrm{GL}(6,\mathbb{R})} &= \begin{pmatrix} E^{-1} & -BE \\ 0 & E \end{pmatrix} \\ L_{\mathrm{GL}(6,\mathbb{R})} &= \begin{pmatrix} E^{-1} & 0 \\ 0 & E \end{pmatrix}. \end{aligned} \tag{2.29}$$

Note that the coset representatives L are orthogonal with respect to the metric (2.26), namely $L^T\eta L = \eta$.

The left invariant 1-form $L^{-1}dL \equiv \Gamma$ satisfying $d\Gamma + \Gamma\wedge\Gamma = 0$ turns out to be

$$\Gamma = \begin{pmatrix} EdE^{-1} & -EdBE \\ 0 & E^{-1}dE \end{pmatrix}. \tag{2.30}$$

As usual we can decompose the left invariant 1-form into the connection Ω, plus the vielbein $\mathcal{P}$:

$$\Gamma = \Omega^H T_H + \mathcal{P}^K T_K\,. \tag{2.31}$$

The matrices T_H are the generators of the isotropy group $\mathrm{SO}(6)_1\times\mathrm{SO}(6)_2$, where we have indicated with $\mathrm{SO}(6)_1 \sim \mathrm{SU}(4)$ the semisimple part of the R-symmetry group U(4) and with $\mathrm{SO}(6)_2$ the "matter group".

Since we are also interested in the connection of the diagonal subgroup $\mathrm{SO}(6)_{(d)}$, we will use in the following two different basis for the generators, the first one that makes explicit the direct product structure of the isotropy group (Cartan basis) and the latter in which we identify the diagonal subgroup of the two factors (diagonal basis). We have respectively:

$$T_H = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix} \qquad T'_H = \begin{pmatrix} T_{(v)} & T_{(a)} \\ T_{(a)} & T_{(v)} \end{pmatrix} \tag{2.32}$$

where $T_{(v)}$ is the generator of the diagonal $\mathrm{SO}(6)_{(d)}$ of $\mathrm{SO}(6)_1\times\mathrm{SO}(6)_2$ and $T_{(a)}$ is the generator of the orthogonal complement. The two basis are related by

$$T'_H = D^{-1}T_H D \tag{2.33}$$

where D is the matrix:

$$D = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{2.34}$$

In the diagonal basis we can extract the connections $\omega^{(d)}$ and $\widehat{\omega}$ of the diagonal $\mathrm{SO}(6)_{(d)}$ subgroup and of its orthogonal part by tracing with the T'_H generators or, more simply, by decomposing $L^{-1}dL$ into its antisymmetric part, giving the connection, and its symmetric part giving the vielbein. In the following we will write Ω and $\mathcal{P}$ as follows:

$$\begin{aligned}\Omega &= \omega^{(d)} + \widehat{\omega} \\ \omega^{(d)} &= \frac{1}{2}\begin{pmatrix} EdE^{-1} - dE^{-1}\,E & 0 \\ 0 & EdE^{-1} - dE^{-1}\,E \end{pmatrix} \\ \widehat{\omega} &= \frac{1}{2}\begin{pmatrix} 0 & -EdBE \\ -EdBE & 0 \end{pmatrix}\end{aligned} \tag{2.35}$$

The vielbein $\mathcal{P}$ is, by definition $\mathcal{P} = \Gamma - \Omega$ so that we get

$$\Omega = \begin{pmatrix} \omega^{IJ} & -P^{[IJ]} \\ -P^{[IJ]} & \omega^{IJ} \end{pmatrix}; \qquad \mathcal{P} = \begin{pmatrix} P^{(IJ)} & -P^{[IJ]} \\ P^{[IJ]} & -P^{(IJ)} \end{pmatrix} \tag{2.36}$$

where

$$\omega^{IJ} = \frac{1}{2}(EdE^{-1} - dE^{-1}\,E)^{IJ} \tag{2.37}$$

$$P^{(IJ)} = \frac{1}{2}(EdE^{-1} + dE^{-1}\,E)^{IJ} \tag{2.38}$$

$$P^{[IJ]} = \frac{1}{2}(EdBE)^{IJ}\,. \tag{2.39}$$

In particular:

$$\nabla^{(d)} E^I_{\ \Lambda} \equiv dE^I_{\ \Lambda} - E^I_{\Lambda}\omega_I^{\ \ J} = -E^J_{\ \Lambda} P^{(JI)}\,. \tag{2.40}$$

In this basis the Maurer-Cartan equation

$$d\Gamma + \Gamma\wedge\Gamma = 0 \tag{2.41}$$

take the form:

$$R^{(d)IJ} = -P^{(IK)}\wedge P^{(KJ)} \tag{2.42}$$

$$\nabla^{(d)} P^{[IJ]} = -P^{(IK)}\wedge P^{[KJ]} + P^{[IK]}\wedge P^{(KJ)} \tag{2.43}$$

$$\nabla^{(d)} P^{(IJ)} = 0 \tag{2.44}$$

where $\nabla^{(d)}$ is the $\mathrm{SO}(6)_d$ covariant derivative and $R^{(d)}$ is the $\mathrm{SO}(6)_d$ curvature:

$$R^{(d)\,IJ} = d\omega^{IJ} + \omega^{IK}\wedge\omega_K^{\ \ J}\,. \tag{2.45}$$

The usual Cartan basis (T_H-basis) where the connection is block-diagonal and the vielbein is block off-diagonal is obtained by rotating Γ with the matrix D. We find:

$$\Gamma = \begin{pmatrix} \omega_1^{IJ} & -(P^{IJ})^T \\ -P^{IJ} & \omega_2^{IJ} \end{pmatrix} = \begin{pmatrix} \omega^{IJ} - P^{[IJ]} & P^{(IJ)} + P^{[IJ]} \\ P^{(IJ)} - P^{[IJ]} & \omega^{IJ} + P^{[IJ]} \end{pmatrix} \tag{2.46}$$

where ω_1 and ω_2 are the connections of $\mathrm{SO}(6)_1$ and $\mathrm{SO}(6)_2$ respectively, while $-(P^{IJ})^T = P^{(IJ)} + P^{[IJ]}$ is the vielbein. In this case the the curvature of the $\mathrm{SO}(6,6)/\mathrm{SO}(6)\otimes\mathrm{SO}(6)$ manifold takes the form:

$$\mathcal{R} = \begin{pmatrix} R_1 & 0 \\ 0 & R_2 \end{pmatrix} \tag{2.47}$$

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where:

$$R_1^{IJ} \equiv d\omega_1^{IJ} + \omega_1^{IK} \wedge \omega_{1\,K}{}^J = -P^{KI} \wedge P_K{}^J \tag{2.48}$$
$$R_2^{IJ} \equiv d\omega_2^{IJ} + \omega_2^{IK} \wedge \omega_{2\,K}{}^J = -P^{IK} \wedge P^J{}_K \tag{2.49}$$

and the vanishing torsion equation is

$$\nabla P^{IJ} \equiv dP^{IJ} + P^{IK} \wedge \omega_{1K}^{\ \ J} + \omega_{2\ K}^{\ I} \wedge P^{KJ} = 0\,. \tag{2.50}$$

2.2 Geometry of the σ-model in presence of n $D3$-branes

We now introduce additional n Yang-Mills multiplets $(A^i_\mu,\, \lambda^i_A,\, \lambda^{Ai},\, q^i_I)$, $I = 1\dots 6$, $i = 1\dots n$.

The isometry group is now $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SO}(6,6+n)$ and the coset representative $\mathbb{L}$ factorizes in the product of the $\frac{\mathrm{SO}(6,6+n)}{\mathrm{SO}(6)\times\mathrm{SO}(6+n)}$ coset representative L and the $\frac{\mathrm{SL}(2,\mathbb{R})}{\mathrm{SO}(2)}$ coset representative S:

$$\mathbb{L} = S\,L\,. \tag{2.51}$$

In the following we shall characterize the matrix form of the various $\mathrm{SO}(6,6+n)$ generators in the $\mathbf{12}+\mathbf{n}$ and define the embedding of $\mathrm{SU}(1,1)\otimes\mathrm{SO}(6,6+n)$ inside $\mathrm{Sp}(24+2n,\mathbb{R})$.

With respect to the subgroup $\mathrm{SL}(6,\mathbb{R})\times\mathrm{SO}(1,1)\times\mathrm{SO}(n)$ the $\mathrm{SO}(6,6+n)$ generators decompose as follows:

$$\mathfrak{so}(6,6+n) = \mathfrak{sl}(6,\mathbb{R})^0 + \mathfrak{so}(1,1)^0 + \mathfrak{so}(n)^0 + (\mathbf{15}',\mathbf{1})^{+2} + (\mathbf{15},\mathbf{1})^{-2} + (\mathbf{6}',\mathbf{n})^{+1} + (\mathbf{6},\mathbf{n})^{-1} \tag{2.52}$$

where the superscript refers to the $\mathfrak{so}(1,1)$ grading. Let us choose for the $\mathbf{12}+\mathbf{n}$ invariant metric η the following matrix:

$$\eta = \begin{pmatrix} 0_{6\times 6} & \mathbb{1}_{6\times 6} & 0_{6\times n} \\ \mathbb{1}_{6\times 6} & 0_{6\times 6} & 0_{6\times n} \\ 0_{n\times 6} & 0_{n\times 6} & -\mathbb{1}_{n\times n} \end{pmatrix} \tag{2.53}$$

where the blocks are defined by the decomposition of the $\mathbf{12}+\mathbf{n}$ into $\mathbf{6}+\mathbf{6}+\mathbf{n}$. The generators in the right hand side of (2.52) have the following form:

$$\mathfrak{sl}(6,\mathbb{R}) : \begin{pmatrix} A & 0 & 0 \\ 0 & -A^T & 0 \\ 0 & 0 & 0 \end{pmatrix}; \qquad \mathfrak{so}(1,1) : \begin{pmatrix} \mathbb{1} & 0 & 0 \\ 0 & -\mathbb{1} & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.54}$$

$$(\mathbf{15}',\mathbf{1})^{+2} : T_{[\Lambda\Sigma]} = \begin{pmatrix} 0 & t_{[\Lambda\Sigma]} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \qquad (\mathbf{15},\mathbf{1})^{-2} : (T_{[\Lambda\Sigma]})^T \tag{2.55}$$

$$(\mathbf{6}',\mathbf{n})^{+1} : T_{(\Lambda i)} = \begin{pmatrix} 0 & 0 & t_{(\Lambda i)} \\ 0 & 0 & 0 \\ 0 & (t_{(\Lambda i)})^T & 0 \end{pmatrix}; \qquad (\mathbf{6},\mathbf{n})^{-1} : (T_{(\Lambda i)})^T \tag{2.56}$$

where we have used the following notation:

$$t_{[\Lambda\Sigma]}{}^{\Gamma\Delta} = \delta^{\Gamma\Delta}_{\Lambda\Sigma}\,; \qquad t_{(\Lambda i)}{}^{\Sigma k} = \delta^\Sigma_\Lambda \delta^k_i \qquad \Lambda,\Sigma = 1,\dots,6\,; \qquad i,k = 1,\dots,n\,. \tag{2.57}$$

As in the preceding case, we split the scalar fields into those which span the $\frac{\mathrm{GL}(6,\mathbb{R})}{\mathrm{SO}(6)_d}$ submanifold and which parametrize the corresponding coset representative $L_{\mathrm{GL}(6,\mathbb{R})}$ from the axions parametrizing the $(\mathbf{15}',\mathbf{1})^{+2}$ translations and we indicate them as before respectively with E^I_Λ and $B^{\Lambda\Sigma}$. In presence of $D3$-branes we have in addition the generators in the $(\mathbf{6}',\mathbf{n})^{+1}$ that we parametrize with the $6\times n$ matrices $a \equiv a^\Lambda_i$ (in the following we will also use the notation $q^I_i \equiv E^I_\Lambda a^\Lambda_i$). The coset representatives $L_{\mathrm{GL}(6,\mathbb{R})}$ and L can thus be constructed as follows:

$$
\begin{aligned}
L = \exp\left(-B^{\Lambda\Sigma}T_{[\Lambda\Sigma]} + a^{\Lambda i}T_{(\Lambda i)}\right)L_{\mathrm{GL}(6,\mathbb{R})} &= \begin{pmatrix} E^{-1} & -CE & a \\ 0 & E & 0 \\ 0 & a^T E & \mathbb{1} \end{pmatrix} \\
L_{\mathrm{GL}(6,\mathbb{R})} &= \begin{pmatrix} E^{-1} & 0 & 0 \\ 0 & E & 0 \\ 0 & 0 & \mathbb{1} \end{pmatrix} \\
C &= B - \frac{1}{2}aa^T
\end{aligned}
\tag{2.58}
$$

where the sum over repeated indices is understood. Note that the coset representative L is orthogonal respect the metric η.

The left invariant 1-form $\Gamma = L^{-1}dL$ turns out to be:

$$
\Gamma = \begin{pmatrix} E\,dE^{-1} & -E[dB - \frac{1}{2}(da\,a^T - a\,da^T)]E & E\,da \\ \mathbf{0}_{6\times 6} & E^{-1}dE & \mathbf{0}_{6\times n} \\ \mathbf{0}_{n\times 6} & da^T\,E & \mathbb{1}_{n\times n} \end{pmatrix}. \tag{2.59}
$$

Proceeding as before we can extract, from the left invariant 1-form, the connection and the vielbein (2.31) in the basis where we take the diagonal subgroup $\mathrm{SO}(6)_d$ inside $\mathrm{SO}(6)\times \mathrm{SO}(6+n)$, where now T_H are the generators of $\mathrm{SO}(6)_1\times\mathrm{SO}(6)_2\times\mathrm{SO}(n)$. It is sufficient to take the antisymmetric and symmetric part of Γ corresponding to the connection and the vielbein respectively. We find:

$$
\Omega = \begin{pmatrix} \omega^{IJ} & -P^{[IJ]} & P^{Ii} \\ -P^{[IJ]} & \omega^{IJ} & -P^{Ii} \\ -P^{iI} & P^{iI} & 0 \end{pmatrix}; \qquad \mathcal{P} = \begin{pmatrix} P^{(IJ)} & -P^{[IJ]} & P^{Ii} \\ P^{[IJ]} & -P^{(IJ)} & P^{Ii} \\ P^{iI} & P^{iI} & 0 \end{pmatrix} \tag{2.60}
$$

where

$$
\omega^{IJ} = \frac{1}{2}(EdE^{-1} - dE^{-1}\,E)^{IJ} \tag{2.61}
$$

$$
P^{(IJ)} = \frac{1}{2}(EdE^{-1} + dE^{-1}\,E)^{IJ} \tag{2.62}
$$

$$
P^{[IJ]} = \frac{1}{2}\left\{E\left[dB - \frac{1}{2}\left(da\,a^T - a\,da^T\right)\right]E\right\}^{IJ} \tag{2.63}
$$

$$
P^{Ii} = \frac{1}{2}E^I_\Lambda da^{\Lambda i}\,. \tag{2.64}
$$

From the Maurer-Cartan equations

$$
d\Gamma + \Gamma\wedge\Gamma = 0 \tag{2.65}
$$

we derive the expression of the curvatures and the equations expressing the absence of torsion in the diagonal basis:

$$
\begin{aligned}
\nabla^{(d)} P^{[IJ]} &= -P^{(IK)} \wedge P^{[KJ]} + P^{[IK]} \wedge P^{(KJ)} + 2P^{I}_{\ i} \wedge P^{iJ} && (2.66)\\
\nabla^{(d)} P^{iJ} &= P^{i}_{\ I} \wedge P^{IJ} && (2.67)
\end{aligned}
$$

while equations (2.42), (2.44) remain unchanged.

Note that, as it is apparent from equation (2.60), the connection of $\mathrm{SO}(n)$ is zero in this gauge: $\omega^{ij} = 0$. To retrieve the form of the connection in the Cartan basis it is sufficient to rotate Ω and $\mathcal{P}$, given in equation (2.60), by the generalized D matrix (2.34)

$$
D = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} . \qquad (2.68)
$$

We find

$$
\Omega = \left(\begin{array}{c|cc} \omega^{IJ} - P^{[IJ]} & \mathbf{0}_{6\times 6} & \mathbf{0}_{6\times n} \\ \hline \mathbf{0}_{6\times 6} & \omega^{IJ} + P^{[IJ]} & P^{Ii} \\ \mathbf{0}_{n\times 6} & -P^{Ii} & \mathbf{0}_{n\times n} \end{array} \right) \qquad (2.69)
$$

$$
\mathcal{P} = \left(\begin{array}{c|cc} \mathbf{0}_{6\times 6} & P^{(IJ)} + P^{[IJ]} & P^{Ii} \\ \hline P^{(IJ)} - P^{[IJ]} & \mathbf{0}_{6\times 6} & \mathbf{0}_{6\times n} \\ P^{iI} & \mathbf{0}_{6\times n} & \mathbf{0}_{n\times n} \end{array} \right) \qquad (2.70)
$$

$$
\begin{aligned}
R_1^{IJ} &\equiv d\omega_1^{IJ} + \omega_1^{IK} \wedge \omega_{1\,K}{}^{J} = -P^{KI} \wedge P_{K}{}^{J} - 2P^{Ii} \wedge P^{iJ} && (2.71)\\
R_2^{IJ} &\equiv d\omega_2^{IJ} + \omega_2^{IK} \wedge \omega_{2\,K}{}^{J} = -P^{IK} \wedge P^{J}{}_{K} + 2P^{Ii} \wedge P^{iJ} && (2.72)
\end{aligned}
$$

and the vanishing torsion equation is

$$
\begin{aligned}
\nabla P^{IJ} \equiv dP^{IJ} + P^{IK} \wedge \omega_{1K}^{\ J} + \omega_2^{IK} \wedge P_{K}{}^{J} + 2P^{Ii} \wedge P_i^{J} &= 0 && (2.73)\\
dP^{Ii} + \omega_1^{IJ} \wedge P_J{}^{i} + P^{(IJ)} \wedge P_J{}^{i} &= 0\,. && (2.74)
\end{aligned}
$$

3. The symplectic embedding and duality rotations

Let us now discuss the embedding of the isometry group $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SO}(6,6+n)$ inside $\mathrm{Sp}(24+2n,\mathbb{R})$. We start from the embedding in which the $\mathrm{SO}(6,6+n)$ is diagonal:[4]

$$
\begin{aligned}
&\mathrm{SO}(6,6+n) \overset{\iota}{\hookrightarrow} \mathrm{Sp}(24+2n,\mathbb{R}) \\
&\qquad g \in \mathrm{SO}(6,6+n) \xrightarrow{\ \iota\ } \iota(g) = \begin{pmatrix} g & 0 \\ 0 & (g^{-1})^{T} \end{pmatrix} \in \mathrm{Sp}(24+2n,\mathbb{R}) \\
&\qquad S = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R}) \xrightarrow{\ \iota\ } \iota(S) = \begin{pmatrix} \alpha \mathbb{1} & -\beta\eta \\ -\gamma\eta & \delta\mathbb{1} \end{pmatrix} \in \mathrm{Sp}(24+2n,\mathbb{R}) \\
&\qquad \alpha\delta - \beta\gamma = 1 \qquad (3.1)
\end{aligned}
$$

[4]The signs in the embedding ι of $\mathrm{SL}(2,\mathbb{R})$ have been chosen in such a way that the action on the doublet charges in the final embedding ι' were the same as S.

where each block of the symplectic matrices is a $(12+n)\times(12+n)$ matrix. In this embedding a generic symplectic section has the following grading structure with respect to $\mathfrak{so}(1,1)$:

$$V_{\rm Sp} = \begin{pmatrix} v^{(+1)} \\ v^{(-1)} \\ v^{(0)} \\ u^{(-1)} \\ u^{(+1)} \\ u^{(0)} \end{pmatrix} \tag{3.2}$$

where $v^{(\pm1)}$ and $u^{(\pm1)}$ are six dimensional vectors while $v^{(0)}$ and $u^{(0)}$ have dimension n. Identifying the v's with the electric field strengths and the u's with their magnetic dual, we note that the embedding ι (3.1) corresponds to the standard embedding where $\mathrm{SL}(2,\mathbb{R})$ acts as electric-magnetic duality while $\mathrm{SO}(6,6+n)$ is purely electric.

We are interested in defining an embedding ι' in which the generators in the $(\mathbf{15}',\mathbf{1})^{+2}$ act as nilpotent off diagonal matrices or Peccei-Quinn generators and the $\mathrm{SL}(2,\mathbb{R})$ group has a block diagonal action on the $v^{(\pm1)}$ and $u^{(\pm1)}$ components and an off diagonal action on the $v^{(0)}$ and $u^{(0)}$ components.

Indeed, our aim is to gauge (at most) twelve of the fifteen translation generators in the representation $(\mathbf{15}',\mathbf{1})^{+2}$ and a suitable subgroup $G\subset \mathrm{SO}(n)$.

The symplectic transformation $\mathcal{O}$ which realizes this embedding starting from the one in (3.1) is easily found by noticing that $(v^{(+1)},\,u^{(+1)})$ and $(v^{(-1)},\,u^{(-1)})$ transform in the $(\mathbf{6}',\mathbf{2})^{+1}$ and $(\mathbf{6},\mathbf{2})^{-1}$ with respect to $\mathrm{GL}(6,\mathbb{R})\times\mathrm{SL}(2,\mathbb{R})$ respectively. Therefore we define the new embedding:

$$\begin{aligned} \iota' &= \mathcal{O}\iota\mathcal{O}^{-1} \\ \mathcal{O} &= \begin{pmatrix} 0 & 0 & 0 & \mathbb{1}_{6\times6} & 0 & 0 \\ 0 & \mathbb{1}_{6\times6} & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbb{1}_{m\times m} & 0 & 0 & 0 \\ -\mathbb{1}_{6\times6} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbb{1}_{6\times6} & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbb{1}_{m\times m} \end{pmatrix} \end{aligned} \tag{3.3}$$

In this embedding the generic $\mathrm{SL}(2,\mathbb{R})$ element S has the following form:

$$\iota'(S) = \begin{pmatrix} \delta\,\mathbb{1}_{6\times6} & -\gamma\,\mathbb{1}_{6\times6} & 0 & 0 & 0 & 0 \\ -\beta\,\mathbb{1}_{6\times6} & \alpha\,\mathbb{1}_{6\times6} & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha\,\mathbb{1}_{m\times m} & 0 & 0 & \beta\,\mathbb{1}_{m\times m} \\ 0 & 0 & 0 & \alpha\,\mathbb{1}_{6\times6} & \beta\,\mathbb{1}_{6\times6} & 0 \\ 0 & 0 & 0 & \gamma\,\mathbb{1}_{6\times6} & \delta\,\mathbb{1}_{6\times6} & 0 \\ 0 & 0 & \gamma\,\mathbb{1}_{m\times m} & 0 & 0 & \delta\,\mathbb{1}_{m\times m} \end{pmatrix} \tag{3.4}$$

while the generic element of $\mathrm{SO}(6,6+n)/\mathrm{SO}(6)\times\mathrm{SO}(6+n)$ takes the form

$$\iota'(L)=\begin{pmatrix} E & 0 & 0 & 0 & 0 & 0\\ 0 & E & 0 & 0 & 0 & 0\\ 0 & a^TE & \mathbb{1} & 0 & 0 & 0\\ 0 & CE & -a & E^{-1} & 0 & 0\\ -CE & 0 & 0 & 0 & E^{-1} & -a\\ -a^TE & 0 & 0 & 0 & 0 & \mathbb{1}\end{pmatrix}. \tag{3.5}$$

The product $\Sigma=\iota'(L)\iota'(S)$ of these two matrices gives the desired embedding in $\mathrm{Sp}(24+2n,\mathbb{R})$ of the relevant coset. If we write the $\mathrm{Sp}(24+2n,\mathbb{R})$ matrix in the form

$$\Sigma=\begin{pmatrix} A & B\\ C & D\end{pmatrix} \tag{3.6}$$

and define

$$f=\frac{1}{\sqrt{2}}(A-iB)\,;\qquad h=\frac{1}{\sqrt{2}}(C-iD) \tag{3.7}$$

we obtain [58]

$$f=\frac{1}{2}\begin{pmatrix} \delta E & -\gamma E & 0\\ -\beta E & \alpha E & 0\\ -\beta a^TE & \alpha a^TE & \alpha-i\beta\end{pmatrix} \tag{3.8}$$

$$h=\frac{1}{2}\begin{pmatrix} -\beta CE-i\alpha E^{-1} & \alpha CE-i\beta E^{-1} & -(\alpha-i\beta)a\\ -\delta CE-i\gamma E^{-1} & \gamma CE-i\delta E^{-1} & -(\gamma-i\delta)a\\ -\delta a^TE & \gamma a^TE & \gamma-i\delta\end{pmatrix}. \tag{3.9}$$

The kinetic matrix of the vectors is defined as [51, 58] $\mathcal{N}=h\cdot f^{-1}$ and we find

$$\mathcal{N}=\begin{pmatrix} -2iL^1\overline{L}^1E^{-1}E^{-1} & \frac{1}{2}aa^T-2iL^{(1}\overline{L}^{2)}E^{-1}E^{-1}-2iBL^{[1}\overline{L}^{2]} & -a\\ \frac{1}{2}aa^T-2iL^{(2}\overline{L}^{1)}E^{-1}E^{-1}-2iBL^{[2}\overline{L}^{1]} & -2iL^2\overline{L}^2E^{-1}E^{-1}+\frac{L^2}{L^1}aa^T & -\frac{L^2}{L^1}a\\ -a^T & -\frac{L^2}{L^1}a^T & \frac{L^2}{L^1}\end{pmatrix} \tag{3.10}$$

or in components

$$\begin{aligned}\mathcal{N}^{\Lambda\alpha\Sigma\beta} &= -2iL^{(\alpha}\overline{L}^{\beta)}(E^{-1})^{\Lambda}{}_{I}(E^{-1})^{I\Sigma}+B^{\Lambda\Sigma}\epsilon^{\alpha\beta}-i(aa^T)^{\Lambda\Sigma}\,L^{(\alpha}\left(\overline{L}^{\beta)}-L^{\beta)}\frac{\overline{L}^1}{L^1}\right)\\ \mathcal{N}^{\Lambda\alpha i} &= -a^{\Lambda i}\frac{L^\alpha}{L^1}\\ \mathcal{N}^{ij} &= \frac{L^2}{L^1}\delta^{ij}\end{aligned} \tag{3.11}$$

where we have used the relation (2.9).

4. The gauging

Our aim is to gauge a group of the following form:

$$T_{12}\times G\subset \mathrm{SO}(6,6+n) \tag{4.1}$$

where T_{12} denote 12 of the $(\mathbf{15}',\mathbf{1})^{+2}$ Peccei-Quinn translations $T_{[\Lambda\Sigma]}$ in SO(6,6) and the group G is in general a compact semisimple subgroup of SO(n) of dimension n. In particular if $G = \mathrm{U}(N)$ we must have $N^2 = n$. The gauge group is a subgroup of the global symmetry group of the ungauged action whose algebra, for the choice of the symplectic embedding defined in the previous section, is:

$$\mathfrak{sl}(6,\mathbb{R})^0 + \mathfrak{so}(1,1)^0 + \mathfrak{so}(n)^0 + (\mathbf{15}',\mathbf{1})^{+2} + (\mathbf{6}',\mathbf{n})^{+1}\,. \tag{4.2}$$

We note that the maximal translation group T_{12} which can be gauged is of dimension twelve since the corresponding gauge vector fields are $A_{\Lambda\alpha}$ belong to the $(\mathbf{6},\mathbf{2})^{-1}$ of $\mathrm{GL}(6,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$. Let us denote the gauge generators of the T_{12} factor by $T^{\Lambda\alpha}$, corresponding to the gauge vectors $A_{\Lambda\alpha}$ and by T^i ($i = 1,\dots,n$) those of the G factor associated with the vectors A_i. These two sets of generators are expressed in terms of the $(\mathbf{15}',\mathbf{1})^{+2}$ generators $T_{[\Lambda\Sigma]}$ and of the SO(n) generators $T_{[ij]}$ respectively by means of suitable *embedding matrices* $f^{\Gamma\Sigma\Lambda\alpha}$ and c^{kij}:

$$\begin{aligned} T^{\Lambda\alpha} &= f^{\Gamma\Sigma\Lambda\alpha}\,T_{[\Gamma\Sigma]} \\ T^k &= c^{kij}\,T_{[ij]} \end{aligned} \tag{4.3}$$

where c^{ijk} are the structure constants of G, with $i\,j\,k$ completely antisymmetric. The constants $f^{\Gamma\Sigma\Lambda\alpha}$ are totally antisymmetric in $\Gamma\Sigma\Lambda$ as a consequence both of supersymmetry and gauge invariance or, in our approach, of the closure of the Bianchi identities. They transform therefore with respect to $\mathrm{SL}(6,\mathbb{R}) \times \mathrm{SO}(1,1) \times \mathrm{SL}(2,\mathbb{R})$ in the $(\mathbf{20}',\mathbf{2})^{+3}$. Note that $f^{\Gamma\Sigma\Lambda\alpha}$ are the remnants in $D = 4$ of the fluxes of the type-IIB three-forms.

We may identify the scalar fields of the theory with the elements of the coset representative L of $\mathrm{SO}(6,6+n)/\mathrm{SO}(6) \times \mathrm{SO}(6+n)$ namely, E_I^Λ, $B^{\Lambda\Sigma}$, a_i^Λ. The scalar field associated with the coset $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2) \simeq \mathrm{SU}(1,1)/\mathrm{U}(1)$ is instead represented by the complex 2-vector L^α satisfying the constraint (2.9).

The gauging can be performed in the usual way replacing the coordinates differentials with the gauge covariant differentials $\nabla_{(g)}$:

$$dL^\alpha \longrightarrow \nabla_{(g)}L^\alpha \equiv dL^\alpha \tag{4.4}$$

$$dE_I^\Lambda \longrightarrow \nabla_{(g)}E_I^\Lambda \equiv dE_I^\Lambda \tag{4.5}$$

$$dB^{\Lambda\Sigma} \longrightarrow \nabla_{(g)}B^{\Lambda\Sigma} = dB^{\Lambda\Sigma} + f^{\Lambda\Sigma\Gamma\alpha}A_{\Gamma\alpha} \tag{4.6}$$

$$da_i^\Lambda \longrightarrow \nabla_{(g)}a_i^\Lambda = da_i^\Lambda + c_i{}^{jk}A_j a_k^\Lambda\,. \tag{4.7}$$

Note that $f^{\Lambda\Sigma\Gamma\alpha}$ are the constant components of the translational Killing vectors in the chosen coordinate system, namely $k^{\Lambda\Sigma|\Gamma\alpha} = f^{\Lambda\Sigma\Gamma\alpha}$ [57], where the couple $\Lambda\Sigma$ are coordinate indices while $\Gamma\alpha$ are indices in the adjoint representation of the gauge subgroup T_{12}; in the same way the Killing vectors of the compact gauge subgroup G are given by $k_i^{\Lambda\,|j} = c_i{}^{jk}a_k^\Lambda$ where the couple Λi are coordinate indices, while jk are in the adjoint representation of G.

From equations (4.4)–(4.7) we can derive the structure of the gauged left-invariant 1-form $\hat{\Gamma}$

$$\hat{\Gamma} = \Gamma + \delta_{(T_{12})}\Gamma + \delta_{(G)}\Gamma \tag{4.8}$$

where $\delta_{(T_{12})}\Gamma$ and $\delta_{(G)}\Gamma$ are the shifts of Γ due to the gauging of T_{12} and G respectively. From these we can compute the shifts of the vielbein and of the connections. We obtain:

$$\begin{aligned}
\hat{P}^{IJ} &= P^{IJ} + \delta_{(T_{12})}P^{IJ} + \delta_G P^{IJ} &(4.9)\\
\hat{P}^{Ii} &= P^{Ii} + \delta_{(T_{12})}P^{Ii} + \delta_G P^{Ii} &(4.10)\\
\hat{\omega}^{IJ}_{1,2} &= \omega_{1,2} + \delta_{(T_{12})}\omega^{IJ}_{1,2} + \delta_G \omega^{IJ}_{1,2} &(4.11)
\end{aligned}$$

where

$$\begin{aligned}
\delta_{(T_{12})}P^{IJ} &= \frac{1}{2}E^I_\Lambda f^{\Lambda\Sigma\Gamma\alpha}A_{\Gamma\alpha}E^J_\Sigma &(4.12)\\
\delta_G P^{IJ} &= \frac{1}{2}c^{ijk}E^I_\Lambda a^\Lambda_i A_j a^\Sigma_k E^J_\Sigma &(4.13)\\
\delta_{(T_{12})}P^{Ii} &= 0 &(4.14)\\
\delta_G P^{Ii} &= \frac{1}{2}c^{ijk}E^I_\Lambda a^\Lambda_k A_j &(4.15)\\
\delta_{(T_{12})}\omega^{IJ}_1 &= -\delta_{(T_{12})}\omega^{IJ}_2 = -\frac{1}{2}E^I_\Lambda f^{\Lambda\Sigma\Gamma\alpha}A_{\Gamma\alpha}E^J_\Sigma &(4.16)\\
\delta_G\omega^{IJ}_1 &= -\delta_G\omega^{IJ}_2 = -\frac{1}{2}c^{ijk}E^I_\Lambda a^\Lambda_i A_j a^\Sigma_k E^J_\Sigma\,. &(4.17)
\end{aligned}$$

Note that only the antisymmetric part of P^{IJ} is shifted, while the diagonal connection $\omega_d = \omega_1 + \omega_2$ remains untouched. An important issue of the gauging is the computation of the "fermion shifts", that is of the extra pieces appearing in the supersymmetry transformation laws of the fermions when the gauging is turned on. Indeed the scalar potential can be computed from the supersymmetry of the lagrangian as a quadratic form in the fermion shifts. The shifts have been computed using the (gauged) Bianchi identities in superspace as it is explained in appendix A. We have:

$$\begin{aligned}
\delta\psi^{(\mathrm{shift})}_{A\mu} &= S_{AB}\gamma_\mu\varepsilon^B = -\frac{i}{48}\left(\overline{F}^{IJK-} + \overline{C}^{IJK-}\right)(\Gamma_{IJK})_{AB}\gamma_\mu\epsilon^B &(4.18)\\
\delta\chi^{A\,(\mathrm{shift})} &= N^{AB}\epsilon_B = -\frac{1}{48}\left(\overline{F}^{IJK+} + \overline{C}^{IJK+}\right)(\Gamma_{IJK})^{AB}\epsilon_B &(4.19)\\
\delta\lambda^{I\,(\mathrm{shift})}_A &= Z^{I\ B}_A\epsilon_B = \frac{1}{8}(F^{IJK} + C^{IJK})(\Gamma_{JK})_A^{\ B}\epsilon_B &(4.20)\\
\delta\lambda^{(\mathrm{shift})}_{iA} &= W_{iA}^{\ \ B}\epsilon_B = \frac{1}{8}L_2\, E^J_\Lambda E^K_\Sigma a^{\Lambda j}a^{\Sigma k}\, c_{ijk}(\Gamma_{JK})_A^{\ B}\epsilon_B &(4.21)
\end{aligned}$$

where we have used the selfduality relation (see appendix D for conventions) $(\Gamma_{IJK})_{AB} = \frac{i}{3!}\epsilon_{IJKLMN}\Gamma^{LMN}_{AB}$ and introduced the quantities

$$\begin{aligned}
F^{\pm\,IJK} &= \frac{1}{2}\left(F^{IJK} \pm i\,{}^*F^{IJK}\right) &(4.22)\\
C^{\pm\,IJK} &= \frac{1}{2}\left(C^{IJK} \pm i\,{}^*C^{IJK}\right) &(4.23)
\end{aligned}$$

where

$$F^{IJK} = L^\alpha f^{IJK}_\alpha\,, \qquad f^{IJK\alpha} = f^{\Lambda\Sigma\Gamma\alpha}E^I_{\ \Lambda}E^J_{\ \Sigma}E^K_{\ \Gamma}\,, \qquad \overline{F}^{IJK} = \overline{L}^\alpha f^{IJK}_\alpha \tag{4.24}$$

and C^{IJK} are the boosted structure constants defined as

$$C^{IJK} = L_2 E^I_\Lambda E^J_\Sigma E^K_\Gamma a^{\Lambda i} a^{\Sigma j} a^{\Gamma k}\, c_{ijk}\,, \qquad \overline{C}^{IJK} = \overline{L}_2 E^I_\Lambda E^J_\Sigma E^K_\Gamma a^{\Lambda i} a^{\Sigma j} a^{\Gamma k}\, c_{ijk} \tag{4.25}$$

while the complex conjugates of the self-dual and antiself-dual components are

$$(F^{\pm IJK})^* = \overline{F}^{\mp IJK}\,, \qquad (C^{\pm IJK})^* = \overline{C}^{\mp IJK} \tag{4.26}$$

For the purpose of the study of the potential, it is convenient to decompose the 24 dimensional representation of $\mathrm{SU}(4)_{(d)} \subset \mathrm{SU}(4)_1 \times \mathrm{SU}(4)_2$ to which λ^I_A belongs, into its irreducible parts, namely $\mathbf{24} = \overline{\mathbf{20}} + \overline{\mathbf{4}}$. Setting:

$$\lambda^I_A = \lambda^{I\,(\overline{20})}_A - \frac{1}{6}(\Gamma^I)_{AB}\lambda^{B(\overline{4})} \tag{4.27}$$

where

$$\lambda^{A(\overline{4})} = (\Gamma_I)^{AB}\lambda^I_B; \qquad (\Gamma_I)^{AB}\lambda^{I\,(\overline{20})}_B = 0 \tag{4.28}$$

we get

$$\delta\lambda^{A(\overline{4})} = Z^{AB(\overline{4})}\epsilon_B = \frac{1}{8}(F^{+IJK} + C^{+IJK})(\Gamma_{IJK})^{AB}\epsilon_B \tag{4.29}$$

$$\delta\lambda^{I\,(\overline{20})}_A = Z^{I(\overline{20})\;B}_A\epsilon_B = \frac{1}{8}(F^{-IJK} + C^{-IJK})(\Gamma_{JK})_A^{\;B}\epsilon_B \tag{4.30}$$

5. Space-time lagrangian

The space-time lagrangian and the associated supersymmetry transformation laws, have been computed using the geometric approach in superspace. We give in the appendices A and B a complete derivation of the main results of this section.

In the following, in order to simplify the notation, we have suppressed the "hats" to the gauged covariant quantities: $\hat{\nabla} \to \nabla$; $\hat{P} \to P$; $\hat{\omega}_{1,2} \to \omega_{1,2}$. In particular, the gauged covariant derivatives on the spinors of the gravitational multiplet and of the Yang-Mills multiplets are defined as follows:

$$\nabla\psi_A = \mathcal{D}\psi_A + \frac{1}{2}q\psi_A - \frac{1}{4}(\Gamma_{IJ})_A^{\;B}\,\omega_1^{IJ}\psi_B \tag{5.1}$$

$$\nabla\chi^A = \mathcal{D}\chi^A + \frac{3}{2}q\chi^A - \frac{1}{4}(\Gamma_{IJ})^A_{\;B}\,\omega_1^{IJ}\chi^B \tag{5.2}$$

$$\nabla\lambda_{IA} = \mathcal{D}\lambda_{IA} - \frac{1}{2}q\lambda_{IA} - \frac{1}{4}(\Gamma_{IJ})_A^{\;B}\,\omega_1^{IJ}\lambda_{IB} + \omega_2^{IJ}\lambda_{JA} \tag{5.3}$$

$$\nabla\lambda_{iA} = \mathcal{D}\lambda_{iA} - \frac{1}{2}q\lambda_{iA} - \frac{1}{4}(\Gamma_{IJ})_A^{\;B}\,\omega_1^{IJ}\lambda_{iB} \tag{5.4}$$

$$\tag{5.5}$$

∇ is the gauged covariant derivative with respect to all the connections that act on the field, while $\mathcal{D}$ is the Lorentz covariant derivative acting on a generic spinor θ as follows

$$\mathcal{D}\theta \equiv d\theta - \frac{1}{4}\omega^{ab}\gamma_{ab}\theta\,. \tag{5.6}$$

The action of the U(1) connection q (2.21) appearing in the covariant derivative ∇ is defined as a consequence of the different U(1) weights of the fields (2.5).

The complete action is:

$$S = \int \sqrt{-g}\mathcal{L}d^4x \tag{5.7}$$

where

$$\mathcal{L} = \mathcal{L}_{(\text{kin})} + \mathcal{L}_{(\text{Pauli})} + \mathcal{L}_{(\text{mass})} - \mathcal{L}_{(\text{potential})} \tag{5.8}$$

where

$$\begin{aligned}
\mathcal{L}_{(\text{kin})} = & -\frac{1}{2}R - i\left(\mathcal{N}^{\Lambda\alpha\Sigma\beta}\mathcal{F}^{+\mu\nu}_{\Lambda\alpha}\mathcal{F}^{+}_{\Sigma\beta\mu\nu} - \overline{\mathcal{N}}^{\Lambda\alpha\Sigma\beta}\mathcal{F}^{-\mu\nu}_{\Lambda\alpha}\mathcal{F}^{-}_{\Sigma\beta\mu\nu}\right) + \\
& -2i\left(\mathcal{N}^{i\Sigma\beta}\mathcal{F}^{+\mu\nu}_{i}\mathcal{F}^{+}_{\Sigma\beta\mu\nu} - \overline{\mathcal{N}}^{i\Sigma\beta}\mathcal{F}^{-\mu\nu}_{i}\mathcal{F}^{-}_{\Sigma\beta\mu\nu}\right) + \\
& -i\left(\mathcal{N}^{ij}\mathcal{F}^{+\mu\nu}_{i}\mathcal{F}^{+}_{j\mu\nu} - \overline{\mathcal{N}}^{ij}\mathcal{F}^{-\mu\nu}_{i}\mathcal{F}^{-}_{j\mu\nu}\right) + \\
& +\frac{2}{3}f^{\Lambda\Sigma\Gamma\gamma}\epsilon^{\alpha\beta}A_{\Gamma\gamma\,\mu}A_{\Sigma\beta\,\nu}F_{\Lambda\alpha\,\rho\sigma}\epsilon^{\mu\nu\rho\sigma} + \overline{p}_\mu p^\mu + \frac{1}{2}P^{IJ}_\mu P^\mu_{IJ} + P^{Ii}_\mu P^\mu_{Ii} + \\
& +\frac{\varepsilon^{\mu\nu\rho\sigma}}{\sqrt{-g}}\left(\bar{\psi}^A_\mu\gamma_\nu\nabla_\rho\psi_{A\sigma} - \bar{\psi}_{A\mu}\gamma_\nu\nabla_\rho\psi^A_\sigma\right) - 2i\left(\bar{\chi}^A\gamma^\mu\nabla_\mu\chi_A + \bar{\chi}_A\gamma^\mu\nabla_\mu\chi^A\right) + \\
& -i\left(\bar{\lambda}^A_I\gamma^\mu\nabla_\mu\lambda^I_A + \bar{\lambda}_{IA}\gamma^\mu\nabla_\mu\lambda^{IA}\right) - 2i\left(\bar{\lambda}^A_i\gamma^\mu\nabla_\mu\lambda^i_A + \bar{\lambda}_{iA}\gamma^\mu\nabla_\mu\lambda^{iA}\right)
\end{aligned} \tag{5.9}$$

where, using equations (2.22), (2.62), (2.63), (2.64) we have:

$$p_\mu\overline{p}^\mu = L_\alpha\overline{L}_\beta\partial_\mu L^\alpha\partial^\mu\overline{L}^\beta \tag{5.10}$$

$$\begin{aligned}
P^{IJ}_\mu P^\mu_{IJ} &= P^{(IJ)}_\mu P^\mu_{(IJ)} + P^{[IJ]}_\mu P^\mu_{[IJ]} \\
P^{(IJ)}_\mu P^\mu_{(IJ)} &= -4\partial_\mu E^I_\Lambda\partial^\mu(E^{-1})^\Lambda_I = 4g_{\Lambda\Sigma}\partial_\mu(E^{-1})^\Lambda_I\partial^\mu(E^{-1})^{I\Sigma} \\
P^{[IJ]}_\mu P^\mu_{[IJ]} &= \frac{1}{4}g_{\Lambda\Gamma}g_{\Sigma\Delta}\left(\nabla_{(g)\mu}B^{\Lambda\Sigma} - \frac{1}{2}a^\Lambda_i\overset{\leftrightarrow}{\nabla}_{(g)\mu}a^{i\Sigma}\right)\left(\nabla^\mu_{(g)}B^{\Gamma\Delta} - \frac{1}{2}a^\Gamma_j\overset{\leftrightarrow}{\nabla}{}^\mu_{(g)}a^{j\Delta}\right) \\
P^{Ij}_\mu P^\mu_{Ij} &= g_{\Lambda\Sigma}\partial_\mu a^\Lambda_i\partial^\mu a^{\Sigma i}
\end{aligned} \tag{5.11}$$

and we have defined $g_{\Lambda\Sigma} \equiv E_{I\Lambda}E^I_\Sigma$, $\mathcal{F}^\pm = \frac{1}{2}(\mathcal{F} \pm i\,{}^*\mathcal{F})$

$$\begin{aligned}
\mathcal{L}_{(\text{Pauli})} = & -2p_\mu\bar{\chi}^A\gamma^\nu\gamma^\mu\psi_{A\nu} - P^{IJ}_\mu\Gamma^{AB}_I\bar{\lambda}_{JA}\gamma^\nu\gamma^\mu\psi_{B\nu} - \\
& -2P^{Ii}_\mu(\Gamma_I)^{AB}\bar{\lambda}_{iA}\gamma^\nu\gamma^\mu\psi_{B\nu} - 2\,\mathrm{Im}\,(\mathcal{N})^{\Lambda\alpha\Sigma\beta}\times \\
& \times\Bigg[\mathcal{F}^{+\mu\nu}_{\Lambda\alpha}\Bigg(L_\beta E_\Sigma{}^I(\Gamma_I)^{AB}\bar{\psi}_{A\mu}\psi_{B\nu} + 2i\overline{L}_\beta E_\Sigma{}^I(\Gamma_I)^{AB}\bar{\chi}_A\gamma_\nu\psi_{B\mu} + \\
& \qquad + 2i\overline{L}_\beta E_\Sigma{}^I\bar{\lambda}^I_A\gamma_\nu\psi^A_\mu + \frac{1}{4}L_\beta E^I_\Sigma(\Gamma_I)_{AB}\bar{\lambda}^A_J\gamma_{\mu\nu}\lambda^{JB} + \overline{L}_\beta E^I_\Sigma\bar{\lambda}^A_I\gamma_{\mu\nu}\chi_A + \\
& \qquad + \frac{1}{2}L_\beta E^I_\Sigma(\Gamma_I)_{AB}\bar{\lambda}^{iA}\gamma_{\mu\nu}\lambda^B_i\Bigg)\Bigg] - 2\,\mathrm{Im}\,(\mathcal{N})^{i\Sigma\beta}\times \\
& \times\Bigg[\mathcal{F}^{+\mu\nu}_i\Bigg(L_\beta E_\Sigma{}^I(\Gamma_I)^{AB}\bar{\psi}_{A\mu}\psi_{B\nu} + 2i\overline{L}_\beta E_\Sigma{}^I(\Gamma_I)^{AB}\bar{\chi}_A\gamma_\nu\psi_{B\mu} + \\
& \qquad + 2i\overline{L}_\beta E^I_\Sigma\bar{\lambda}^I_A\gamma_\nu\psi^A_\mu + \frac{1}{4}L_\beta E^I_\Sigma(\Gamma_I)_{AB}\bar{\lambda}^A_J\gamma_{\mu\nu}\lambda^{JB} + \overline{L}_\beta E^I_\Sigma\bar{\lambda}^A_I\gamma_{\mu\nu}\chi_A +
\end{aligned}$$

$$
\begin{aligned}
& + \frac{1}{2} L_\beta E^I_\Sigma (\Gamma_I)_{AB} \bar{\lambda}^{iA} \gamma_{\mu\nu} \lambda^B_i \Bigg) + \\
& + \mathcal{F}^{+\mu\nu}_{\Sigma\beta} \Bigg(E^I_\Lambda a^\Lambda_i L_2 (\Gamma_I)^{AB} \bar{\psi}_{A\mu} \psi_{B\nu} + 2i E^I_\Lambda a^\Lambda_i \overline{L}_2 (\Gamma_I)^{AB} \bar{\chi}_A \gamma_\nu \psi_{B\mu} + \\
& \quad + 2i E^I_\Lambda a^\Lambda_i \overline{L}_2 \bar{\lambda}_{IA} \gamma_\nu \psi^A_\mu + 4i \overline{L}_2 \bar{\lambda}_{iA} \gamma_\nu \psi^A_\mu + \\
& \quad + \frac{1}{4} L_2 E^I_\Sigma a^\Sigma_i (\Gamma_I)_{AB} \bar{\lambda}^A_J \gamma_{\mu\nu} \lambda^{JB} + \overline{L}_2 E^I_\Sigma a^\Sigma_i \bar{\lambda}^A_I \gamma_{\mu\nu} \chi_A + \\
& \quad + \frac{1}{2} L_2 E^I_\Sigma a^\Sigma_i (\Gamma_I)_{AB} \bar{\lambda}^A_j \gamma_{\mu\nu} \lambda^{jB} + 2 \overline{L}_2 \bar{\lambda}^A_i \gamma_{\mu\nu} \chi_A \Bigg) \Bigg] - \\
& -2 \, \mathrm{Im}\, (\mathcal{N})^{ij} \Bigg[\mathcal{F}^{+\mu\nu}_i \Bigg(E^I_\Lambda a^\Lambda_j L_2 (\Gamma_I)^{AB} \bar{\psi}_{A\mu} \psi_{B\nu} + 2i E^I_\Lambda a^\Lambda_j \overline{L}_2 (\Gamma_I)^{AB} \bar{\chi}_A \gamma_\nu \psi_{B\mu} + \\
& \quad + 2i E^I_\Lambda a^\Lambda_j \overline{L}_2 \bar{\lambda}^I_A \gamma_\nu \psi^A_\mu + 4i \overline{L}_2 \bar{\lambda}_{jA} \gamma_\nu \psi^A_\mu + \\
& \quad + \frac{1}{4} L_2 E^I_\Sigma a^\Sigma_i (\Gamma_I)_{AB} \bar{\lambda}^A_J \gamma_{\mu\nu} \lambda^{JB} + \overline{L}_2 E^I_\Sigma a^\Sigma_i \bar{\lambda}^A_I \gamma_{\mu\nu} \chi_A + \\
& \quad + \frac{1}{2} L_2 E^I_\Sigma a^\Sigma_i (\Gamma_I)_{AB} \bar{\lambda}^A_j \gamma_{\mu\nu} \lambda^{jB} + 2 \overline{L}_2 \bar{\lambda}^A_i \gamma_{\mu\nu} \chi_A \Bigg) \Bigg] + \text{c.c.}
\end{aligned}
\tag{5.12}
$$

$$
\begin{aligned}
\mathcal{L}_{(mass)} = & \Big[+2i S^{AB} \bar{\psi}_{A\mu} \gamma^{\mu\nu} \psi_{B\nu} - 4i N^{AB} \bar{\psi}_{\mu A} \gamma^\mu \chi_B - 2i Z_{IB}{}^A \bar{\psi}_{A\mu} \gamma^\mu \lambda^{IB} - \\
& - 4i W_{iB}{}^A \bar{\psi}_{A\mu} \gamma^\mu \lambda^{iB} + 6 \Big(Q^A_{I\,B} \bar{\lambda}^I_A \chi^B + R^A_{i\,B} \bar{\lambda}^i_A \chi^B + T^{AB}_{IJ} \bar{\lambda}^I_A \lambda^J_B + \\
& + U^{AB}_{ij} \bar{\lambda}^i_A \lambda^j_B + + V^{AB}_{Ij} \bar{\lambda}^I_A \lambda^j_B \Big) \Big] + \text{c.c.}
\end{aligned}
\tag{5.13}
$$

$$
\mathcal{L}_{(\text{potential})} = \frac{1}{4} \left(-12 S^{AB} S_{AB} + 4 N^{AB} N_{AB} + 2 Z_{IB}{}^A Z^{IB}{}_A + 4 W_{iB}{}^A W^{iB}{}_A \right). \tag{5.14}
$$

The structures appearing in $\mathcal{L}_{(mass)}$ and $\mathcal{L}_{(potential)}$ are given by

$$
\begin{aligned}
S_{AB} &= -\frac{i}{48} \left(\overline{F}^{IJK-} + \overline{C}^{IJK-} \right) (\Gamma_{IJK})_{AB} && (5.15) \\
N^{AB} &= -\frac{1}{48} \left(\overline{F}^{IJK+} + \overline{C}^{IJK+} \right) (\Gamma_{IJK})^{AB} && (5.16) \\
Z^{I\,B}_A &= \frac{1}{8} \left(F^{IJK} + C^{IJK} \right) (\Gamma_{JK})_A{}^B && (5.17) \\
W_{iA}{}^B &= \frac{1}{8} L_2 \, q^{Jj} q^{Kk} \, c_{ijk} (\Gamma_{JK})_A{}^B && (5.18) \\
Q^{IA}{}_B &= -\frac{1}{12} \left(F^{IJK} + C^{IJK} \right) (\Gamma_{JK})^A{}_B && (5.19) \\
R^{iA}{}_B &= -\frac{1}{24} L_2 \, q^{Jj} q^{Kk} \, c_{ijk} (\Gamma_{JK})^A{}_B && (5.20) \\
T^{IJAB} &= -\frac{1}{3} \delta^{IJ} N^{AB} + \frac{1}{12} \left(\overline{F}^{IJK} + \overline{C}^{IJK} \right) (\Gamma_K)^{AB} && (5.21) \\
U^{ijAB} &= -\frac{2}{3} \delta^{ij} N^{AB} - \frac{1}{3} \overline{L}_2 q^I_k c^{ijk} (\Gamma_I)^{AB} && (5.22)
\end{aligned}
$$

$$V^{IiAB} = -\frac{1}{3}c^{ijk}\overline{L}_2 q^I_j q^K_k (\Gamma_K)^{AB}\,. \tag{5.23}$$

The lagrangian is invariant under the following supersymmetry transformation laws:

$$\begin{aligned}
\delta V^a_\mu &= -i\bar{\psi}_{A\mu}\gamma^a\varepsilon^A + \text{c.c.}\\
\delta A_{\Lambda\alpha\mu} &= -L_\alpha E^I_\Lambda(\Gamma_I)^{AB}\bar{\psi}_{A\mu}\varepsilon_B + iL_\alpha E^I_\Lambda(\Gamma_I)_{AB}\bar{\chi}^A\gamma_\mu\varepsilon^B +\\
&\quad + iL_\alpha E_{\Lambda\,I}\bar{\lambda}^{IA}\gamma_\mu\varepsilon_A + c.c.\\
\delta A_{i\mu} &= -L_2 E^I_\Lambda a^\Lambda_i(\Gamma_I)^{AB}\bar{\psi}_{A\mu}\varepsilon_B + iL_2 E^I_\Lambda a^\Lambda_i(\Gamma_I)_{AB}\bar{\chi}^A\gamma_\mu\varepsilon^B +\\
&\quad + iL_2 E_{\Lambda I}a^\Lambda_i\bar{\lambda}^{IA}\gamma_\mu\varepsilon_A + 2iL_2\bar{\lambda}^A_i\gamma_\mu\varepsilon_A + c.c.\\
p_\beta\delta L^\beta &\equiv = 2\bar{\chi}_A\varepsilon^A \Longrightarrow \delta L^\alpha = 2\overline{L}^\alpha\bar{\chi}_A\varepsilon^A\\
\delta^{(I}_K E^{J)}_\Lambda\delta\left(E^{-1}\right)^{\Lambda K} &= -(\Gamma^{(I})^{AB}\bar{\lambda}^{J)}_A\varepsilon_B + \text{c.c.}\\
\frac{1}{2}E^I_\Lambda E^J_\Sigma\left(\delta B^{\Lambda\Sigma} - \delta a^{[\Lambda}_i a^{\Sigma]i}\right) &= (\Gamma^{[I})^{AB}\bar{\lambda}^{J]}_A\varepsilon_B + \text{c.c.}\\
\frac{1}{2}E^I_\Lambda\delta a^{\Lambda i} &= (\Gamma^I)^{AB}\bar{\lambda}^i_A\varepsilon_B + \text{c.c.}\\
\delta\psi_{\mu A} &= \mathcal{D}_\mu\varepsilon_A - \overline{L}^\alpha(E^{-1})^\Lambda_I(\Gamma^I)_{AB}\mathcal{F}^-_{\Lambda\alpha|\mu\nu}\gamma^\nu\varepsilon^B + S_{AB}\gamma_\mu\varepsilon^B\\
\delta\chi^A &= \frac{i}{2}\overline{p}_\mu\gamma^\mu\varepsilon^A + \frac{i}{4}\overline{L}^\alpha(E^{-1})^\Lambda_I(\Gamma^I)^{AB}\mathcal{F}^-_{\Lambda\alpha|\mu\nu}\gamma^{\mu\nu}\varepsilon_B + N^{AB}\varepsilon_B\\
\delta\lambda_{IA} &= \frac{i}{2}(\Gamma^J)_{AB}P_{JI|\,\mu}\gamma^\mu\varepsilon^B - \frac{i}{2}L^\alpha(E^{-1})^\Lambda_I\mathcal{F}^-_{\Lambda\alpha|\mu\nu}\gamma^{\mu\nu}\varepsilon_A + Z_{IA}{}^B\varepsilon_B\\
\delta\lambda_{iA} &= \frac{i}{2}(\Gamma^J)_{AB}P_{Ji|\,\mu}\gamma^\mu\varepsilon^B + \frac{1}{4}\frac{1}{\overline{L}_2}\mathcal{F}^-_{i|\mu\nu}\gamma^{\mu\nu}\varepsilon_A -\\
&\quad - \frac{1}{4}\frac{1}{\overline{L}_2}a^\Lambda_i\mathcal{F}^-_{2\Lambda|\mu\nu}\gamma^{\mu\nu}\varepsilon_A + W_{iA}{}^B\varepsilon_B
\end{aligned} \tag{5.24}$$

where we have defined

$$q^I_i = E^I_\Lambda a^\Lambda_I\,. \tag{5.25}$$

6. The scalar potential and its extrema

From the expression of the potential given in the lagrangian (5.14) and using the fermionic shifts given in equations (5.15)–(5.18), one obtains that the potential is given by a sum of two terms, each being a square modulus, namely:

$$V = \frac{1}{12}\left|F^{IJK-} + C^{IJK-}\right|^2 + \frac{1}{8}\left|L_2 c_{ijk}q^{jJ}q^{kK}\right|^2 \tag{6.1}$$

where F^{IJK-} and C^{IJK-} were defined in equations (4.22) and (4.23) and q^I_i in equation (5.25).

In the first term F^{IJK-} represents the contribution to the potential of the bulk fields, while C^{IJK-} is the contribution from the D-branes sector. On the other hand the second term is the generalization of the potential already present in the super Yang-Mills theory [50].

We note that using the decomposition given in equations (4.29),(4.30), namely:

$$Z^{I\,B}_A = Z^{I\,B(\overline{20})}_A - \frac{1}{6}(\Gamma^I)_A{}^C Z^{(\overline{4})}_{CB} \tag{6.2}$$

in equation (5.14), the contribution of the gravitino shift S_{AB} cancels exactly against the contribution $Z^{(4)}_{AB}$ of the representation $\overline{\mathbf{4}}$ of the gaugino; furthermore, since $Z^{I\ B(\overline{20})}_{A}$ is proportional to N_{AB}, the bulk part of the potential is proportional to $N_{AB}N^{BA}$.

We now discuss the extrema of this potential. Since V is positive semidefinite, its extrema are given by the solution of $V=0$, that is

$$L_2 c_{ijk} q^{jJ} q^{kK} = 0\,, \qquad F^{IJK-} + C^{IJK-} = 0\,. \tag{6.3}$$

In absence of fluxes $F^{IJK}=0$, the only solution is that the q^{jJ} belong to the Cartan subalgebra of G, then $C^{IJK}=0$, but all the moduli of the orientifold are not stabilized as well as the L_α. In this case the moduli space is

$$\frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)} \times \frac{\mathrm{SO}(6,6+N)}{\mathrm{SO}(6)\times \mathrm{SO}(6+N)} \tag{6.4}$$

where N is the dimension of the Cartan subalgebra of G.

In presence of fluxes, $F^{IJK}\neq 0$ we have, besides q^{jJ} belonging to the Cartan subalgebra of G, also $F^{IJK-}=0$. We now show that this condition freezes the dilaton field S and several moduli of $\mathrm{GL}(6)/\mathrm{SO}(6)_d$. We note that the equation $F^{IJK-}=0$ can be rewritten as:

$$L^1 f_1^{IJK-} + L^2 f_2^{IJK-} = 0 \Longrightarrow \frac{f_1^{IJK-}}{f_2^{IJK-}} = -\frac{L^2}{L^1} \equiv S \tag{6.5}$$

so that S must be a constant. We set:

$$S = i\alpha \to \frac{L^2}{L^1} = -i\alpha \Longrightarrow \frac{\phi_2}{\phi_1} = \frac{1-\alpha}{1+\alpha} \tag{6.6}$$

where α is a complex constant and $\mathrm{Re}\,\alpha \equiv e^{\varphi_0} > 0$ Equation (6.5) becomes:

$$f_1^{\Lambda\Sigma\Delta} = +\mathrm{Re}\,\alpha\,{}^*f_2^{\Lambda\Sigma\Delta} - \mathrm{Im}\,\alpha f_2^{\Lambda\Sigma\Delta}\,. \tag{6.7}$$

We rewrite this equation using $f_{1,2}^{IJK-} = \frac{1}{2}\left(f_{1,2}^{IJK} - i\,{}^*f_{1,2}^{IJK}\right)$ and by replacing the SO(6) indices I,J,K with GL(6) indices via the coset representatives E^I_Λ; we find:

$$f_1^{\Lambda\Sigma\Gamma} + \mathrm{Im}\,\alpha f_2^{\Lambda\Sigma\Gamma} = \frac{1}{3!}\,\mathrm{Re}\,\alpha\ det E^{-1} \epsilon^{\Lambda\Sigma\Gamma\Delta\Pi\Omega} g_{\Delta\Delta'} g_{\Pi\Pi'} g_{\Omega\Omega'} f_2^{\Delta'\Pi'\Omega'} \tag{6.8}$$

where $g^{\Lambda\Sigma} \equiv E^\Lambda_I E^{\Sigma I}$ is the (inverse) moduli metric of T^6 in $\mathrm{GL}(6)/\mathrm{SO}(6)_d$.

It is convenient to analyze equation (6.8) using the complex basis defined in appendix C. In this basis only 4 fluxes (together with their complex conjugates) corresponding to the eigenvalues of the gravitino mass matrix, are different from zero.

Going to the complex basis where each Greek index decomposes as $\Lambda = (i,\bar{\imath})$, $i = 1,2,3$ [57] and lowering the indices on both sides, one obtains an equation relating a (p,q) form on the l.h.s ($p,q = 0,1,2,3$; $p+q=3$) to a combination of (p',q') forms on the r.h.s. Requiring that all the terms with $p'\neq p$, $q'\neq q$ are zero and that the r.h.s. of equation (6.8) be a constant, on is led to fix different subsets of the $g^{\Lambda\Sigma}$ moduli, depending on the residual degree of supersymmetry.

Suppose now that we have $N = 3$ unbroken supersymmetry, that is $m_1 = m_2 = m_3 = 0$, $m_4 \propto |f^{123}| \neq 0$ (see appendix C). The previous argument, concerning (p,q)-forms, allows us to conclude that all the components g_{ij} and $g_{\bar{\imath}\bar{\jmath}}$ are zero, so that at the $N = 3$ minimum we have:

$$g_{\Lambda\Sigma} \longrightarrow \begin{pmatrix} g_{i\bar{\jmath}} & 0 \\ 0 & g_{\bar{\imath}j} \end{pmatrix}. \tag{6.9}$$

In the $N = 2$ case we have $m_2 = m_3 = 0$, $m_1 \propto |f^{1\overline{23}}| \neq 0$, $m_4 \propto |f^{123}| \neq 0$ and a careful analysis of equation (6.8) shows that, besides the previous frozen moduli, also the $g_{1\bar{2}}$, $g_{1\bar{3}}$ components are frozen.

Finally, in the $N = 1$ case we set one of the masses equal to zero, say $m_2 = 0$, and $m_1 \propto |f^{1\overline{23}}| \neq 0$, $m_3 \propto |f^{12\bar{3}}| \neq 0$ $m_4 \propto |f^{123}| \neq 0$; in this case the only surviving moduli are the diagonal ones, namely $g_{i\bar{\imath}}$, so that, using the results given in appendix C, also the real components of $g_{\Lambda\Sigma}$ are diagonal

$$g_{\Lambda\Sigma} \longrightarrow \mathrm{diag}\left\{e^{2\varphi_1},\, e^{2\varphi_2},\, e^{2\varphi_3},\, e^{2\varphi_1},\, e^{2\varphi_2},\, e^{2\varphi_3}\right\} \tag{6.10}$$

the exponentials representing the radii of the manifold $T^2_{(14)} \times T^2_{(25)} \times T^2_{(36)}$. In terms of the vielbein E^I_Λ we have:

$$E^I_\Lambda = \mathrm{diag}(e^{\varphi_1}, e^{\varphi_2}, e^{\varphi_3}, e^{\varphi_1}, e^{\varphi_2}, e^{\varphi_3})\,. \tag{6.11}$$

Finally, when all the masses are different from zero ($N = 0$), no further condition on the moduli is obtained.

Note that in every case equation (6.8) reduces to the $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{GL}(6,\mathbb{R})$ non covariant constraint among the fluxes:

$$f_1^{\Lambda\Sigma\Gamma} + \mathrm{Im}\,\alpha f_2^{\Lambda\Sigma\Gamma} = \frac{1}{3!}\,\mathrm{Re}\,\alpha\ \epsilon^{\Lambda\Sigma\Gamma\Delta\Pi\Omega} f_2^{\Delta\Pi\Omega}\,. \tag{6.12}$$

In the particular case $\alpha = 1$, which implies $\varphi = C = 0$, from equations (6.5), (6.6), we obtain:

$$f_1^{-\Lambda\Sigma\Delta} = i f_2^{-\Lambda\Sigma\Delta}\,. \tag{6.13}$$

In this case the minimum of the scalar potential is given by

$$\phi_2 = 0 \Longrightarrow |\phi_1| = 1 \tag{6.14}$$

or, in terms of the L^α fields, $L^1 = \frac{1}{\sqrt{2}}$, $L^2 = -\frac{i}{\sqrt{2}}$ (up to an arbitrary phase) . Furthermore (6.7) reduces to the duality relation:

$$f_1{}^{\Lambda\Sigma\Delta} = \frac{1}{3!}\epsilon^{\Lambda\Sigma\Delta\Gamma\Pi\Omega} f_2{}^{\Gamma\Pi\Omega} \tag{6.15}$$

which is the $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{GL}(6)$ non covariant constraint imposed in [43].

Let us now discuss the residual moduli space in each case. For this purpose we introduce, beside the metric $g^{\Lambda\Sigma}$, also the 15 axions $B^{\Lambda\Sigma}$ with enlarge $\mathrm{GL}(6)/\mathrm{SO}(6)_d$ to $\mathrm{SO}(6,6)/\mathrm{SO}(6) \times \mathrm{SO}(6)$ (using complex coordinates $B^{\Lambda\Sigma} = (B^{i\bar{\jmath}},\ B^{ij} = -B^{ji})$.

Since we have seen that for $N = 1, 0$, the frozen moduli from $F^{IJK-} = 0$ are all the $g^{i\bar{\jmath}}$ and g^{ij} except the diagonal ones $g^{i\bar{\imath}}$, correspondingly, in the $B^{\Lambda\Sigma}$ sector, all B^{ij} and

$B^{i\bar{j}}$ are frozen, except the diagonal $B^{i\bar{i}}$, and they are eaten by the 12 bosons through the Higgs mechanism. Indeed the three diagonal $B^{i\bar{i}}$ are inert under gauge transformation (see appendix C). The metric moduli space is $(\mathrm{O}(1,1))^3$ which, adding the axions, enlarges to the coset space $(\mathrm{U}(1,1)/\mathrm{U}(1) \times \mathrm{U}(1))^3$. Adding the Yang-Mills moduli in the $D3$-brane sector, the full moduli space of a generic vacuum with completely broken supersymmetry, or $N = 1$ supersymmetry contains $6+6N$ moduli which parametrize three copies of $\mathrm{U}(1,1+N)/\mathrm{U}(1) \times \mathrm{U}(1+N)$.

Let us consider now the situation of partial supersymmetry breaking (for a more detailed discussion see appendix C). For $N = 3$ supersymmetry the equation $F^{IJK-} = 0$ freezes all g^{ij} moduli but none of the $g^{i\bar{j}}$. The relevant moduli space of metric $g^{i\bar{j}}$ is nine dimensional and given by $\mathrm{GL}(3,\mathbb{C})/\mathrm{U}(3)$. Correspondingly there are six massive vectors whose longitudinal components are the B^{ij} axions. Adding the nine uneaten $B^{i\bar{j}}$ axions the total moduli space is $\mathrm{U}(3,3)/\mathrm{U}(3) \times \mathrm{U}(3)$. Further adding the $6N$ Cartan moduli, the complete moduli space is $\mathrm{U}(3,3+N)/\mathrm{U}(3) \times \mathrm{U}(3+N)$.

For $N = 2$ unbroken supersymmetry the equation $F^{IJK-} = 0$ fixes all g^{ij} and $g^{i\bar{j}}$ except the diagonal ones and $g^{2\bar{3}}$. The moduli space of the metric is $\mathrm{SO}(1,1) \times \mathrm{GL}(2,\mathbb{C})/\mathrm{U}(2)$. There are 10 massive vectors which eat all B^{ij} moduli and $B^{i\bar{j}}$ except the diagonal ones and $B^{2\bar{3}}$; the complete moduli space enlarges to $\mathrm{SU}(1,1)/\mathrm{U}(1)\times\mathrm{SU}(2,2)/\mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{U}(1)$. This space is the product of the one-dimensional Kähler manifold and the two-dimensional quaternionic manifold as required by $N = 2$ supergravity. By further adding the $6N$ Cartan moduli, the moduli space enlarges to $\mathrm{SU}(1,1+N)/\mathrm{U}(1) \times \mathrm{SU}(1+N) \times \mathrm{SU}(2,2+N)/\mathrm{SU}(2) \times \mathrm{SU}(2+N) \times \mathrm{U}(1)$.

Finally in the case of $N = 1$ unbroken supersymmetry, the frozen moduli are the same as in the $N = 0$ case, and the moduli space is indeed the product of three copies of Kähler-Hodge manifolds, as appropriate to chiral multiplets.

7. The mass spectrum

The spectrum of this theory contains 128 states (64 bosons and 64 fermions) coming from the bulk states of IIB supergravity and $16N^2$ states coming from the n $D3$-branes. The brane sector is $N = 4$ supersymmetric. Setting $\alpha = 1$, the bulk part has a mass spectrum which has a surprisingly simple form.

In units of the overall factor $\frac{\sqrt{2}}{24}e^{\frac{K}{2}}$, $K = 2\varphi_1 + 2\varphi_2 + 2\varphi_3$ being the Kähler potential of the moduli space $\left(\frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)}\right)^3$, one finds:[5]

Fermions			Bosons		
$(4)\,spin\,\frac{3}{2}$	$\lvert m_i\rvert$	$i = 1,2,3,4$	$(12)\,spin\,1$	$\lvert m_i \pm m_j\rvert$	$i < j$
$2(4)\,spin\,\frac{1}{2}$	$\lvert m_i\rvert$		$(6)\,spin\,0$	$m = 0$	
$(16)\,spin\,\frac{1}{2}$	$\lvert m_i \pm m_j \pm m_k\rvert$	$i < j < k$	$(12)\,spin\,0$	$\lvert m_i \pm m_j\rvert$	
			$(8)\,spin\,0$	$\lvert m_1 \pm m_2 \pm m_3 \pm m_4\rvert$	

[5]Note that we have corrected a mistake in the spin 1 mass formula (5.21)-(5.23) as given in reference [57]

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where m_i, $i = 1,\ldots 4$ is the modulus of the complex eigenvalues of the matrix $f_1^{IJK-}(\Gamma_{IJK})_{AB}$ evaluated at the minimum.

Note that in the case $\alpha \neq 1$ all the masses m_i acquire an α-dependent extra factor due to the relation

$$\overline{F}^{IJK-} = -\sqrt{2}(\mathrm{Re}\,\alpha)^{\frac{1}{2}} f_2^{IJK-} = i\sqrt{2}\frac{(\mathrm{Re}\,\alpha)^{\frac{1}{2}}}{\alpha} f_1^{IJK-} \tag{7.1}$$

so that all the spectrum is rescaled by a factor $g(\alpha) \equiv \sqrt{2}\frac{(\mathrm{Re}\,\alpha)^{\frac{1}{2}}}{\alpha}$.

This spectrum is identical (in suitable units) to a truncation (to half of the states) of the mass spectrum of the $N = 8$ spontaneously broken supergravity à la Scherk-Schwarz [35]–[37].

The justification of this statement in given in the next section.

We now note some properties of the spectrum. For arbitrary values of m_i the spectrum satisfies the quadratic and quartic mass relations

$$\sum_J (2J+1)(-1)^{2J} m_J^{2k} = 0 \qquad k = 1, 2\,. \tag{7.2}$$

Note that, in proving the above relations, the mixed terms for $k = 1$ are of the form $m_i m_j$ and they separately cancel for bosons and fermions, due to the symmetry $m_i \longrightarrow -m_i$ of the spectrum. On the other hand, for $k = 2$ the mixed terms $m_i^2 m_j^2$ are even in m_i and thus cancel between bosons and fermions.

If we set some of the $m_i = 0$ we recover the spectrum of $N = 3$, 2, 1, 0 supersymmetric phases.

If we set $|m_i| = |m_j|$ for some i, j we recover some unbroken gauge symmetries. This is impossible with the $N = 3$ phase (when $m_2 = m_3 = m_4 = 0$) but it is possible in the $N \leq 2$ phases. For instance in the $N = 2$ phase, for $m_3 = m_4 = 0$ and $|m_1| = |m_2|$ there is an additional massless vector multiplet, while in the $N = 1$ phase, for $m_4 = 0$, $|m_1| = |m_2| = |m_3|$ there are three massless vector multiplets and finally for the $N = 0$ phase and all $|m_i|$ equal, there are six massless vectors.

The spectrum of the $D3$-brane sector has an enhanced ($N = 4$) supersymmetry, so when the gauge group is spontaneously broken to its Cartan subalgebra $\mathrm{U}(N) \longrightarrow \mathrm{U}(1)^N$, $N(N-1)$ charged gauge bosons become massive and they are $\frac{1}{2}$ BPS saturated multiplets of the $N = 4$ superalgebra with central charges (the fermionic sector for the brane gaugini is discussed at the end of section 8). The residual N Cartan multiplets remain massless and their scalar partners complete the $6 + 6N$ dimensional moduli space of the theory, that is classically given by three copies of $\frac{\mathrm{SU}(1,1+N)}{\mathrm{U}(1)\times\mathrm{SU}(1,1+N)}$.

Adding all these facts together we may say that the spectrum is classified by the following quantum numbers (q, e_i), where q are "charges" of the bulk gauge group, namely $|m_i|$, $|m_i \pm m_j|$, $|m_i \pm m_j \pm m_k|$, $|m_i \pm m_j \pm m_k \pm m_\ell|$ and e_i are the $N - 1$ charges of the $\mathrm{SU}(N)$ root-lattice. In the supergravity spectrum there is a sector of the type $(q, 0)$ (the 128 states coming from the bulk) and a sector of the type $(0, e_i)$ (the sector coming from the $D3$-brane).

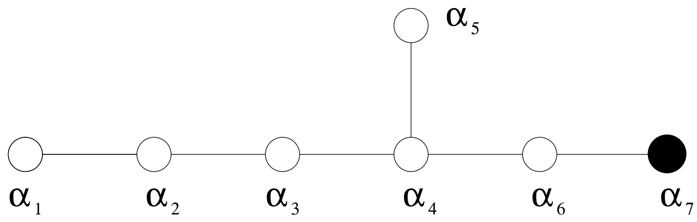

Figure 1: $E_{7(7)}$ Dynkin diagram. The empty circles denote $SO(6,6)_T$ roots, while the filled circle denotes the $SO(6,6)_T$ spinorial weight.

8. Embedding of the $N = 4$ model with six matter multiplets in the $N = 8$

There are two inequivalent ways of embedding the $N = 4$ model with an action which is invariant under global $SL(2,\mathbb{R}) \times GL(6,\mathbb{R})$, within the $N = 8$ theory. They correspond to the two different embeddings of the $SL(2,\mathbb{R}) \times GL(6,\mathbb{R})$ symmetry of the $N = 4$ action inside $E_{7(7)}$ which is the global symmetry group of the $N = 8$ field equations and Bianchi identities.

The $N = 8$ model describes the low energy limit of type-II superstring theory compactified on a six torus T^6. As shown in [60, 61, 62] the ten dimensional origin of the 70 scalar fields of the model can be characterized group theoretically once the embeddings of the isometry group $SO(6,6)_T$ of the moduli space of T^6 and of the duality groups of higher dimensional maximal supergravities are specified within $E_{7(7)}$. This analysis makes use of the *solvable Lie algebra representation* which consists in describing the scalar manifold as a solvable group manifold generated by a solvable Lie algebra of which the scalar fields are the parameters. The solvable Lie algebra associated with $E_{7(7)}$ is defined by its Iwasawa decomposition and is generated by the seven Cartan generators and by the 63 shift generators corresponding to all the positive roots. In this representation the Cartan subalgebra is parametrized by the scalars coming from the diagonal entries of the internal metric while all the other scalar fields are in one to one correspondence with the $E_{7(7)}$ positive roots. We shall represent the $E_{7(7)}$ Dynkin diagram as in figure 1. The positive roots are expressed as combinations $\alpha = \sum_{i=1}^{7} n^i \alpha_i$ of the simple roots in which the positive integers n^i define the *grading* of the root α with respect to α_i. The isometry group of the T^6 moduli space $SO(6,6)_T$ is defined by the sub-Dynkin diagram $\{\alpha_i\}_{i=1,\dots,6}$ while the Dynkin diagram of the duality group $E_{11-D(11-D)}$ of the maximal supergravity in dimension $D > 4$ is obtained from the $E_{7(7)}$ Dynkin diagram by deleting the simple roots $\{\alpha_1, \dots, \alpha_{D-4}\}$. Using these conventions in table 1 [62] the correspondence between the 63 non dilatonic scalar fields deriving from dimensional reduction of type-IIB theories and positive roots of $E_{7(7)}$ is illustrated.

IIB	ϵ_i-components	n^i gradings
$C^{(0)}$	$\frac{1}{2}(-1,-1,-1,-1,-1,-1,\sqrt{2})$	$(0,0,0,0,0,0,1)$
$\mathcal{B}_{5\,6}$	$(0,0,0,0,1,1,0)$	$(0,0,0,0,0,1,0)$
$g_{5\,6}$	$(0,0,0,0,1,-1,0)$	$(0,0,0,0,1,0,0)$

Table 1: Continued.

IIB	ϵ_i-components	n^i gradings
$C_{5\,6}$	$\frac{1}{2}(-1,-1,-1,-1,1,1,\sqrt{2})$	$(0,0,0,0,0,1,1)$
$\mathcal{B}_{4\,5}$	$(0,0,0,1,1,0,0)$	$(0,0,0,1,1,1,0)$
$g_{4\,5}$	$(0,0,0,1,-1,0,0)$	$(0,0,0,1,0,0,0)$
$\mathcal{B}_{4\,6}$	$(0,0,0,1,0,1,0)$	$(0,0,0,1,0,1,0)$
$g_{4\,6}$	$(0,0,0,1,0,-1,0)$	$(0,0,0,1,1,0,0)$
$C_{4\,5}$	$\frac{1}{2}(-1,-1,-1,1,1,-1,\sqrt{2})$	$(0,0,0,1,1,1,1)$
$C_{4\,6}$	$\frac{1}{2}(-1,-1,-1,1,-1,1,\sqrt{2})$	$(0,0,0,1,0,1,1)$
$\mathcal{B}_{3\,4}$	$(0,0,1,1,0,0,0)$	$(0,0,1,2,1,1,0)$
$g_{3\,4}$	$(0,0,1,-1,0,0,0)$	$(0,0,1,0,0,0,0)$
$\mathcal{B}_{3\,5}$	$(0,0,1,0,1,0,0)$	$(0,0,1,1,1,1,0)$
$g_{3\,5}$	$(0,0,1,0,-1,0,0)$	$(0,0,1,1,0,0,0)$
$\mathcal{B}_{3\,6}$	$(0,0,1,0,0,1,0)$	$(0,0,1,1,0,1,0)$
$g_{3\,6}$	$(0,0,1,0,0,-1,0)$	$(0,0,1,1,1,0,0)$
$C_{3\,4}$	$\frac{1}{2}(-1,-1,1,1,-1,-1,\sqrt{2})$	$(0,0,1,2,1,1,1)$
$C_{3\,5}$	$\frac{1}{2}(-1,-1,1,-1,1,-1,\sqrt{2})$	$(0,0,1,1,1,1,1)$
$C_{3\,6}$	$\frac{1}{2}(-1,-1,1,-1,-1,1,\sqrt{2})$	$(0,0,1,1,0,1,1)$
$C_{3\,4\,5\,6}$	$\frac{1}{2}(-1,-1,1,1,1,1,\sqrt{2})$	$(0,0,1,2,1,2,1)$ $\leftarrow$
$\mathcal{B}_{2\,3}$	$(0,1,1,0,0,0,0)$	$(0,1,2,2,1,1,0)$
$g_{2\,3}$	$(0,1,-1,0,0,0,0)$	$(0,1,0,0,0,0,0)$
$\mathcal{B}_{2\,4}$	$(0,1,0,1,0,0,0)$	$(0,1,1,2,1,1,0)$
$g_{2\,4}$	$(0,1,0,-1,0,0,0)$	$(0,0,1,0,0,0,0)$
$\mathcal{B}_{2\,5}$	$(0,1,0,0,1,0,0)$	$(0,1,1,1,1,1,0)$
$g_{2\,5}$	$(0,1,0,0,-1,0,0)$	$(0,0,0,1,0,0,0)$
$\mathcal{B}_{2\,6}$	$(0,1,0,0,0,1,0)$	$(0,1,1,1,0,1,0)$
$g_{2\,6}$	$(0,1,0,0,0,-1,0)$	$(0,0,0,0,1,0,0)$
$C_{2\,3}$	$\frac{1}{2}(-1,1,1,-1,-1,-1,\sqrt{2})$	$(0,1,2,2,1,1,1)$
$C_{2\,4}$	$\frac{1}{2}(-1,1,-1,1,-1,-1,\sqrt{2})$	$(0,1,1,2,1,1,1)$
$C_{2\,5}$	$\frac{1}{2}(-1,1,-1,-1,1,-1,\sqrt{2})$	$(0,1,1,1,1,1,1)$
$C_{2\,6}$	$\frac{1}{2}(-1,1,-1,-1,-1,1,\sqrt{2})$	$(0,1,1,1,0,1,1)$
$C_{2\,4\,5\,6}$	$\frac{1}{2}(-1,1,-1,1,1,1,\sqrt{2})$	$(0,1,1,2,1,2,1)$ $\leftarrow$
$C_{2\,3\,5\,6}$	$\frac{1}{2}(-1,1,1,-1,1,1,\sqrt{2})$	$(0,1,2,2,1,2,1)$ $\leftarrow$
$C_{2\,3\,4\,6}$	$\frac{1}{2}(-1,1,1,1,-1,1,\sqrt{2})$	$(0,1,2,3,1,2,1)$ $\leftarrow$
$C_{2\,3\,4\,5}$	$\frac{1}{2}(-1,1,1,1,1,-1,\sqrt{2})$	$(0,1,2,3,1,2,1)$ $\leftarrow$
$\mathcal{B}_{1\,2}$	$(1,1,0,0,0,0,0)$	$(1,2,2,2,1,1,0)$
$g_{1\,2}$	$(1,-1,0,0,0,0,0)$	$(1,0,0,0,0,0,0)$
$\mathcal{B}_{1\,3}$	$(1,0,1,0,0,0,0)$	$(1,1,2,2,1,1,0)$
$g_{1\,3}$	$(1,0,-1,0,0,0,0)$	$(1,1,0,0,0,0,0)$
$\mathcal{B}_{1\,4}$	$(1,0,0,1,0,0,0)$	$(1,1,1,2,1,1,0)$
$g_{1\,4}$	$(1,0,0,-1,0,0,0)$	$(1,1,1,0,0,0,0)$

Table 1: Continued.

IIB	ϵ_i-components	n^i gradings
$\mathcal{B}_{15}$	$(1,0,0,0,1,0,0)$	$(1,1,1,1,1,1,0)$
g_{15}	$(1,0,0,0,-1,0,0)$	$(1,1,1,1,0,0,0)$
$\mathcal{B}_{16}$	$(1,0,0,0,0,1,0)$	$(1,1,1,1,0,1,0)$
g_{16}	$(1,0,0,0,0,-1,0)$	$(1,1,1,1,1,0,0)$
$\mathcal{B}_{\mu\nu}$	$(0,0,0,0,0,0,\sqrt{2})$	$(1,2,3,4,2,3,2)$
$C_{\mu\nu}$	$\frac{1}{2}(1,1,1,1,1,1,\sqrt{2})$	$(1,2,3,4,2,3,1)$
C_{12}	$\frac{1}{2}(1,1,-1,-1,-1,-1,\sqrt{2})$	$(1,2,2,2,1,1,1)$
C_{13}	$\frac{1}{2}(1,-1,1,-1,-1,-1,\sqrt{2})$	$(1,1,2,2,1,1,1)$
C_{14}	$\frac{1}{2}(1,-1,-1,1,-1,-1,\sqrt{2})$	$(1,1,1,2,1,1,1)$
C_{15}	$\frac{1}{2}(1,-1,-1,-1,1,-1,\sqrt{2})$	$(1,1,1,1,1,1,1)$
C_{16}	$\frac{1}{2}(1,-1,-1,-1,-1,1,\sqrt{2})$	$(1,1,1,1,0,1,1)$
C_{1456}	$\frac{1}{2}(1,-1,-1,1,1,1,\sqrt{2})$	$(1,1,1,2,1,2,1)$ $\leftarrow$
C_{1356}	$\frac{1}{2}(1,-1,1,-1,1,1,\sqrt{2})$	$(1,1,2,2,1,2,1)$ $\leftarrow$
C_{1346}	$\frac{1}{2}(1,-1,1,1,-1,1,\sqrt{2})$	$(1,1,2,3,1,2,1)$ $\leftarrow$
C_{1345}	$\frac{1}{2}(1,-1,1,1,1,-1,\sqrt{2})$	$(1,1,2,3,2,2,1)$ $\leftarrow$
C_{1256}	$\frac{1}{2}(1,1,-1,-1,1,1,\sqrt{2})$	$(1,2,2,2,1,2,1)$ $\leftarrow$
C_{1246}	$\frac{1}{2}(1,1,-1,1,-1,1,\sqrt{2})$	$(1,2,2,3,1,2,1)$ $\leftarrow$
C_{1245}	$\frac{1}{2}(1,1,-1,1,1,-1,\sqrt{2})$	$(1,2,2,3,2,2,1)$ $\leftarrow$
C_{1236}	$\frac{1}{2}(1,1,1,-1,-1,1,\sqrt{2})$	$(1,2,3,3,1,2,1)$ $\leftarrow$
C_{1235}	$\frac{1}{2}(1,1,1,-1,1,-1,\sqrt{2})$	$(1,2,3,3,2,2,1)$ $\leftarrow$
C_{1234}	$\frac{1}{2}(1,1,1,1,-1,-1,\sqrt{2})$	$(1,2,3,4,2,2,1)$ $\leftarrow$

Table 1: Correspondence between the 63 non dilatonic scalar fields from type-IIB string theory on T^6 ($C^{(0)}$, $C^{(2)} \equiv C_{ij}$ and $C^{(4)} \equiv C_{ijkl}$) and positive roots of $\mathrm{E}_{7(7)}$ according to the solvable Lie algebra formalism. The $N = 4$ Peccei-Quinn scalars correspond to roots with grading 1 with respect to β, namely those with $n^6 = 2$ and $n^7 = 1$ which are marked by an arrow in the table.

In this framework the R-R scalars, for instance, are defined by the positive roots which are spinorial with respect to $\mathrm{SO}(6,6)_T$, i.e. which have grading $n^7 = 1$ with respect to the spinorial simple root α_7. On the contrary the NS-NS scalars are defined by the roots with $n^7 = 0,\, 2$.

Let us first discuss the embedding of the $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SO}(6,6)$ duality group of our model within $\mathrm{E}_{7(7)}$. In the solvable Lie algebra language the Peccei-Quinn scalars parametrize the maximal abelian ideal of the solvable Lie algebra generating the scalar manifold. As far as the manifold $\mathrm{SO}(6,6)/\mathrm{SO}(6) \times \mathrm{SO}(6)$ is concerned, this abelian ideal is 15 dimensional and is generated by the shift operators corresponding to positive $\mathrm{SO}(6,6)$ roots with grading one with respect to the simple root placed at one of the two symmetric ends of the corresponding Dynkin diagram D_6. Since in our model the Peccei-Quinn scalars are of R-R type, the $\mathrm{SO}(6,6)$ duality group embedded in $\mathrm{E}_{7(7)}$ does not coincide with $\mathrm{SO}(6,6)_T$. Indeed one of its symmetric ends should be a spinorial root of $\mathrm{SO}(6,6)_T$. Moreover the $\mathrm{SL}(2,\mathbb{R})$ group commuting with $\mathrm{SO}(6,6)$ should coincide with the $\mathrm{SL}(2,\mathbb{R})_{IIB}$ symmetry group of the ten dimensional type-IIB theory, whose Dynkin diagram consists in our formalism of the simple

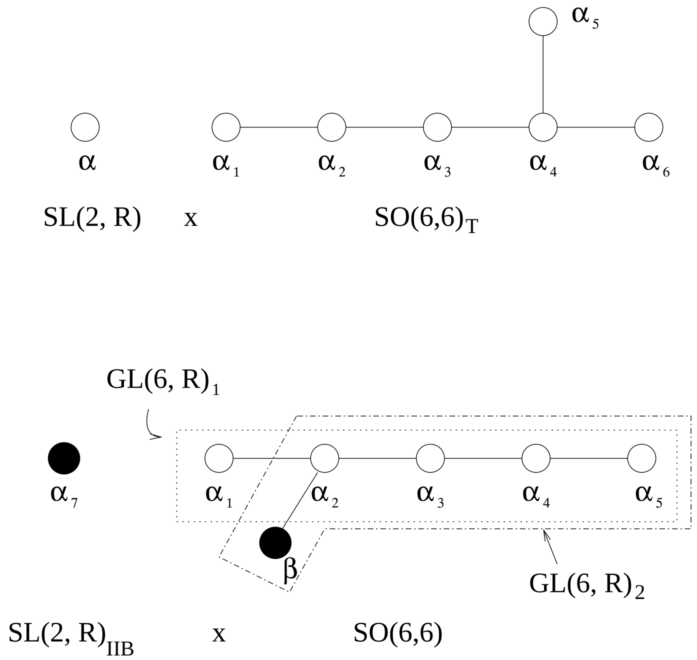

Figure 2: $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SO}(6,6)_T$ and $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SO}(6,6)$ Dynkin diagrams. The root α is the $\mathrm{E}_{7(7)}$ highest root while β is $\alpha_3 + 2\alpha_4 + \alpha_5 + 2\alpha_6 + \alpha_7$. The group $\mathrm{SL}(2,\mathbb{R})_{IIB}$ is the symmetry group of the ten dimensional type-IIB theory.

root α_7. This latter condition uniquely determines the embedding of $\mathrm{SO}(6,6)$ to be the one defined by $\mathrm{D}_6 = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \beta\}$, where $\beta = \alpha_3 + 2\alpha_4 + \alpha_5 + 2\alpha_6 + \alpha_7$ is the spinorial root (see figure 2). On the other hand the 20 scalar fields parametrizing the manifold $\mathrm{SL}(6,\mathbb{R})/\mathrm{SO}(6)$ are all of NS-NS type (they come from the components of the T^6 metric). This fixes the embedding of $\mathrm{SL}(6,\mathbb{R})$ within $\mathrm{E}_{7(7)}$ which we shall denote by $\mathrm{SL}(6,\mathbb{R})_1$: its Dynkin diagram is $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$. The Peccei-Quinn scalars are then defined by the positive roots with grading one with respect to the spinorial end β of D_6 which is not contained in $\mathrm{SL}(6,\mathbb{R})_1$. In table 1 the scalar fields in $\mathrm{SO}(1,1) \times \mathrm{SL}(6,\mathbb{R})_1/\mathrm{SO}(6)$ which are not dilatonic (i.e. do not correspond to diagonal entries of the T^6 metric) correspond to the $\mathrm{SL}(6,\mathbb{R})_1$ positive roots which are characterized by $n^6 = n^7 = 0$ and are the off-diagonal entries of the internal metric. The Peccei-Quinn scalars on the other hand correspond to the roots with grading one with respect to β, which in table 1 are those with $n^6 = 2$, $n^7 = 1$ and indeed, as expected, are identified with the internal components of the type-IIB four form.

The above analysis based on the microscopic nature of the scalars present in our model has led us to select one out of two inequivalent embeddings of the $\mathrm{SL}(6,\mathbb{R})$ group within $\mathrm{E}_{7(7)}$ which we shall denote by $\mathrm{SL}(6,\mathbb{R})_1$ and $\mathrm{SL}(6,\mathbb{R})_2$. The former corresponds to the A_5 Dynkin diagram running from α_1 to α_5 while the latter to the A_5 diagram running from β to α_5. The $\mathrm{SL}(6,\mathbb{R})_1$ symmetry group of our $N = 4$ lagrangian is uniquely defined as part of the maximal subgroup $\mathrm{SL}(3,\mathbb{R}) \times \mathrm{SL}(6,\mathbb{R})_1$ of $\mathrm{E}_{7(7)}$ (in which $\mathrm{SL}(3,\mathbb{R})$ represents an enhancement of $\mathrm{SL}(2,\mathbb{R})_{IIB}$ [60]) with respect to which the relevant $\mathrm{E}_{7(7)}$ representations branch as follows:

$$\mathbf{56} \to (\mathbf{1},\mathbf{20}) + (\mathbf{3}',\mathbf{6}) + (\mathbf{3},\mathbf{6}') \tag{8.1}$$

$$\mathbf{133} \to (\mathbf{8},\mathbf{1}) + (\mathbf{3},\mathbf{15}) + (\mathbf{3}',\mathbf{15}') + (\mathbf{1},\mathbf{35})\,. \tag{8.2}$$

Moreover with respect to the $\mathrm{SO}(3)\times\mathrm{SO}(6)$ subgroup of $\mathrm{SL}(3,\mathbb{R})\times\mathrm{SL}(6,\mathbb{R})_1$ the relevant $\mathrm{SU}(8)$ representations branch in the following way:

$$\begin{aligned}\mathbf{8} &\to (\mathbf{2},\mathbf{4})\\ \mathbf{56} &\to (\mathbf{2},\mathbf{20}) + (\mathbf{4},\overline{\mathbf{4}})\\ \mathbf{63} &\to (\mathbf{1},\mathbf{15}) + (\mathbf{3},\mathbf{1}) + (\mathbf{3},\mathbf{15})\\ \mathbf{70} &\to (\mathbf{1},\mathbf{20}) + (\mathbf{3},\mathbf{15}) + (\mathbf{5},\mathbf{1})\,.\end{aligned} \tag{8.3}$$

The group $\mathrm{GL}(6,\mathbb{R})_2$ on the other hand is contained inside both $\mathrm{SL}(8,\mathbb{R})$ and $\mathrm{E}_{6(6)}\times\mathrm{O}(1,1)$ as opposite to $\mathrm{GL}(6,\mathbb{R})_1$. As a consequence of this it is possible in the $N=8$ theory to choose electric field strengths and their duals in such a way that $\mathrm{SL}(2,\mathbb{R})\times\mathrm{GL}(6,\mathbb{R})_2$ is contained in the global symmetry group of the action while this is not the case for the group $\mathrm{SL}(2,\mathbb{R})\times\mathrm{GL}(6,\mathbb{R})_1\subset\mathrm{SL}(3,\mathbb{R})\times\mathrm{SL}(6,\mathbb{R})_1$. Indeed as it is apparent from eq. (8.1) the electric/magnetic charges in the **56** of $\mathrm{E}_{7(7)}$ do not branch with respect to $\mathrm{SL}(6,\mathbb{R})_1$ into two 28 dimensional reducible representations as it would be required in order for $\mathrm{SL}(6,\mathbb{R})_1$ to be contained in the symmetry group of the lagrangian. On the other hand with respect to the group $\mathrm{SL}(2,\mathbb{R})\times\mathrm{O}(1,1)\times\mathrm{SO}(6)\subset\mathrm{SL}(3,\mathbb{R})\times\mathrm{SL}(6,\mathbb{R})_1$ the **56** branches as follows (the grading as usual refers to the $\mathrm{O}(1,1)$ factor):

$$\mathbf{56} \to (\mathbf{1},\mathbf{10})_0 + \left(\mathbf{1},\overline{\mathbf{10}}\right)_0 + (\mathbf{1},\mathbf{6})_{+2} + (\mathbf{1},\mathbf{6})_{-2} + (\mathbf{2},\mathbf{6})_{+1} + (\mathbf{2},\mathbf{6})_{-1}\,. \tag{8.4}$$

In truncating to the $N=4$ model the charges in the $(\mathbf{1},\mathbf{10})_0+(\mathbf{1},\overline{\mathbf{10}})_0+(\mathbf{1},\mathbf{6})_{+2}+(\mathbf{1},\mathbf{6})_{-2}$ are projected out and the symmetry group of the lagrangian is enhanced to $\mathrm{SL}(2,\mathbb{R})\times\mathrm{GL}(6,\mathbb{R})_1$.

8.1 The masses in the $N=4$ theory with gauged Peccei-Quinn isometries and $\mathrm{USp}(8)$ weights

As we have seen, in the $N=4$ theory with gauged Peccei-Quinn isometries, the parameters of the effective action at the origin of the scalar manifold are encoded in the tensor $f_\alpha{}^{IJK}$. The condition for the origin to be an extremum of the potential, when $\alpha=1$, constrains the fluxes in the following way:

$$f_1{}^{-IJK} - \mathrm{i} f_2{}^{-IJK} = 0 \tag{8.5}$$

therefore all the independent gauge parameters will be contained in the combination $f_1{}^{-IJK}+\mathrm{i}f_2{}^{-IJK}$ transforming in the $\mathbf{10}^{+1}$ with respect to $\mathrm{U}(4)$ and in its complex conjugate which belongs to the $\overline{\mathbf{10}}^{-1}$. Using the gamma matrices each of these two tensors can be mapped into 4×4 symmetric complex matrices:

$$\begin{aligned}B_{AB} &= \left(f_1{}^{-IJK} + \mathrm{i} f_2{}^{-IJK}\right)\Gamma^{IJK}{}_{AB} \;\in\; \mathbf{10}^{+1}\\ \overline{B}^{AB} &= \left(f_1{}^{+IJK} - \mathrm{i} f_2{}^{+IJK}\right)\Gamma_{IJK}{}^{AB} \;\in\; \overline{\mathbf{10}}^{-1}\end{aligned} \tag{8.6}$$

where the matrix B_{AB} is proportional to the gravitino mass matrix S_{AB}. If we denote by $A_A{}^B$ a generic generator of $\mathfrak{u}(4)$ we may formally build the representation of a generic $\mathfrak{usp}(8)$ generator in the **8**:

$$\begin{pmatrix} A_A{}^B & B_{AC} \\ -\overline{B}^{DB} & -A_C{}^D \end{pmatrix} \in \mathfrak{usp}(8)\,. \tag{8.7}$$

The U(1) group in U(4) is generated by $A_A{}^B = \mathrm{i}\,\delta_A{}^B$. Under a U(4) transformation $\mathcal{A}$ the matrix B transforms as follows:

$$B \to \mathcal{A} B \mathcal{A}^t \tag{8.8}$$

Therefore using U(4) transformations the off diagonal generators in the $\mathfrak{usp}(8)/\mathfrak{u}(4)$ can be brought to the following form

$$\begin{aligned} \begin{pmatrix} \mathbf{0} & B^{(d)} \\ -B^{(d)} & \mathbf{0} \end{pmatrix} &\equiv m_i\, H_i \\ B^{(d)} &= \mathrm{diag}(m_1, m_2, m_3, m_4) \quad m_i > 0 \end{aligned} \tag{8.9}$$

where the phases and thus the signs of the m_i were fixed using the $\mathrm{U}(1)^4$ transformations inside U(4) and H_i denote a basis of generators of the $\mathfrak{usp}(8)$ Cartan subalgebra. The gravitino mass matrix represents just the upper off diagonal block of the $\mathfrak{usp}(8)$ Cartan generators in the **8**.

As far as the vectors are concerned we may build the $\mathfrak{usp}(8)$ generators in the **27** in much the same way as we did for the gravitini case, by using the $\mathfrak{u}(4)$ generators in the **15** and in the $\mathbf{6}+\mathbf{6}$ to form the diagonal 15×15 and 12×12 blocks of a 27×27 matrix.

$$\begin{pmatrix} A_{15\times 15} & K_{15\times 12} \\ K_{12\times 15} & A_{12\times 12} \end{pmatrix} \in \mathfrak{usp}(8) \tag{8.10}$$

Here $A_{15\times 15} \equiv A^{\Lambda\Sigma}{}_{\Gamma\Delta}$, $A_{12\times 12} \equiv A^{\Lambda\alpha}{}_{\Gamma\beta}$ while $K_{15\times 12} \equiv K^{\Lambda\Sigma|\Gamma\alpha} = f^{\Lambda\Sigma\Gamma\alpha}$ and $K_{12\times 15} = -K^T_{15\times 12}$.

The vector mass matrix is:

$$M^2_{(\mathrm{vector})} \propto K^t_{15\times 12} K_{15\times 12}\,. \tag{8.11}$$

By acting by means of U(4) on the rectangular matrix $K_{15\times 12}$ it is possible to reduce it to the upper off-diagonal part of a generic element of the $\mathfrak{usp}(8)$ Cartan subalgebra:

$$\begin{aligned} K_{15\times 12} &= \begin{pmatrix} a_1 & 0 & \dots & \dots & 0 \\ 0 & a_2 & 0 & \dots & 0 \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & \vdots \\ 0 & \dots & \dots & 0 & a_{12} \\ 0 & 0 & \dots & \dots & 0 \\ 0 & 0 & \dots & \dots & 0 \\ 0 & 0 & \dots & \dots & 0 \end{pmatrix} \\ a_\ell &= |m_i \pm m_j| \quad 1 \le i < j \le 4 \;\; ; \;\; m_i \ge 0\,. \end{aligned} \tag{8.12}$$

Using equation (8.11) we may read the mass eigenvalues for the vectors which are just a_ℓ.

The above argument may be extended also to the gaugini and the scalars as discussed in the next section.

8.2 Duality with a truncation of the spontaneously broken $N = 8$ theory from Scherk-Schwarz reduction

As discussed in the previous sections the microscopic interpretation of the fields in our $N = 4$ model is achieved by its identification, at the ungauged level, with a truncation of the $N = 8$ theory describing the field theory limit of IIB string theory on T^6. To this end the symmetry group of the $N = 4$ action is interpreted as the $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{GL}(6,\mathbb{R})_1$ inside the $\mathrm{SL}(3,\mathbb{R}) \times \mathrm{SL}(6,\mathbb{R})_1$ maximal subgroup of $\mathrm{E}_{7(7)}$, which is the natural group to consider when interpreting the four dimensional theory from the type-IIB point of view, since the $\mathrm{SL}(3,\mathbb{R})$ factor represents an enhancement of the type-IIB symmetry group $\mathrm{SL}(2,\mathbb{R})_{IIB} \times \mathrm{SO}(1,1)$, where $\mathrm{SO}(1,1)$ is associated to the T^6 volume, while $\mathrm{SL}(6,\mathbb{R})_1$ is the group acting on the moduli of the T^6 metric. A different microscopic interpretation of the ungauged $N = 4$ theory would follow from the identification of its symmetry group with the group $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{GL}(6,\mathbb{R})_2$ contained in both $\mathrm{E}_{6(6)} \times \mathrm{O}(1,1)$ and $\mathrm{SL}(8,\mathbb{R})$ subgroups of $\mathrm{E}_{7(7)}$, where, although the $\mathrm{SL}(2,\mathbb{R})$ factor is still $\mathrm{SL}(2,\mathbb{R})_{IIB}$, the fields are naturally interpreted in terms of dimensionally reduced M-theory since $\mathrm{GL}(6,\mathbb{R})_2$ this time is the group acting on the moduli of the T^6 torus from $D = 11$ to $D = 5$. At the level of the $N = 4$ theory the $\mathrm{SL}(6,\mathbb{R})_1$ and the $\mathrm{SL}(6,\mathbb{R})_2$ are equivalent, while their embedding in $\mathrm{E}_{7(7)}$ is different and so is the microscopic interpretation of the fields in the corresponding theories. Our gauged model is obtained by introducing in the model with $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{GL}(6,\mathbb{R})_1$ manifest symmetry a gauge group characterized by a flux tensor transforming in the $(\mathbf{2},\mathbf{20})^{+3}$. It is interesting to notice that if the symmetry of the ungauged action were identified with $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{GL}(6,\mathbb{R})_2$ formally we would have the same $N = 4$ gauged model, but, as we are going to show, this time we could interpret it as a truncation to $N = 4$ of the spontaneously broken $N = 8$ theory deriving from a Scherk-Schwarz reduction from $D = 5$. The latter, as mentioned in the introduction, is a gauged $N = 8$ theory which is completely defined once we specify the gauge generator $T_0 \in \mathfrak{e}_{6,6}$ to be gauged by the graviphoton arising from the five dimensional metric. The gauging (couplings, masses etc…) is therefore characterized by the $\mathbf{27}$ representation of T_0, namely by the flux matrix $f_r{}^s{}_0$ $(r, s = 1,\ldots,27)$, element of $\mathrm{Adj}(\mathfrak{e}_{6,6}) = \mathbf{78}$ [47]. Decomposing this representation with respect to $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(6,\mathbb{R})_2$ we have:

$$\mathbf{78} \to (\mathbf{3},\mathbf{1}) + (\mathbf{1},\mathbf{35}) + (\mathbf{2},\mathbf{20})\,. \tag{8.13}$$

The representation $(\mathbf{2},\mathbf{20})$ defines the gaugings in which we choose:

$$T_0 \in \frac{\mathfrak{e}_{6,6}}{\mathfrak{sl}(2,\mathbb{R}) + \mathfrak{sl}(6,\mathbb{R})_2}\,. \tag{8.14}$$

These generators can be either compact or non-compact. However, it is known that only for compact T_0 the gauged $N = 8$ theory is a "no-scale" model with a Minkowski vacuum at the origin of the moduli space (flat gaugings). Let us consider the relevant branchings of $\mathrm{E}_{7(7)}$ representations with respect to $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(6,\mathbb{R})_2$:

$$\begin{aligned}
\mathbf{56} &\to (\mathbf{2},\mathbf{6}')_{+1} + (\mathbf{2},\mathbf{6})_{-1} + (\mathbf{1},\mathbf{15}')_{-1} + (\mathbf{1},\mathbf{15})_{+1} + (\mathbf{1},\mathbf{1})_{-3} + (\mathbf{1},\mathbf{1})_{+3}\\
\mathbf{133} &\to (\mathbf{3},\mathbf{1})_0 + (\mathbf{1},\mathbf{1})_0 + (\mathbf{1},\mathbf{35})_0 + (\mathbf{2},\mathbf{20})_0 + (\mathbf{2},\mathbf{6})_{+2} + (\mathbf{2},\mathbf{6}')_{-2} + (\mathbf{1},\mathbf{15}')_{+2} + (\mathbf{1},\mathbf{15})_{-2}
\end{aligned}$$

where the $(\mathbf{2},\mathbf{6})_{-1}+(\mathbf{1},\mathbf{15'})_{-1}$ in the first branching denote the vectors deriving from five dimensional vectors while $(\mathbf{1},\mathbf{1})_{-3}$ is the graviphoton. The truncation to $N=4$ is achieved at the bosonic level by projecting the $\mathbf{56}$ into $(\mathbf{2},\mathbf{6'})_{+1}+(\mathbf{2},\mathbf{6})_{-1}$ and the $\mathbf{133}$ into the adjoint of $\mathrm{SL}(2,\mathbb{R})\times\mathrm{SO}(6,6)$, namely $(\mathbf{3},\mathbf{1})_0+(\mathbf{1},\mathbf{1})_0+(\mathbf{1},\mathbf{35})_0+(\mathbf{1},\mathbf{15'})_{+2}+(\mathbf{1},\mathbf{15})_{-2}$.

If we chose T_0 within $(\mathbf{2},\mathbf{20})$ as a 27×27 generator it has only non vanishing entries $f^{\Lambda\Sigma\Gamma\alpha}$ in the blocks $(\mathbf{1},\mathbf{15'})\times(\mathbf{2},\mathbf{6'})$ and $(\mathbf{2},\mathbf{6'})\times(\mathbf{1},\mathbf{15'})$ and inspection into the couplings of these theories shows that the truncation to $N=4$ is indeed consistent and that we formally get the $N=4$ gauged theory considered in this paper with six matter multiplets. Moreover the extremality condition $f_1{}^{-IJK}-\mathrm{i}f_2{}^{-IJK}=0$ discussed in the previous section coincides with the condition on T_0 to be compact:

$$f_1{}^{-IJK}-\mathrm{i}f_2{}^{-IJK}=0 \quad\Leftrightarrow\quad T_0\in\frac{\mathfrak{usp}(8)}{\mathfrak{so}(2)+\mathfrak{so}(6)}\quad (N=8 \text{ flat gauging})$$

After restricting the $(\mathbf{2},\mathbf{20})$ generators T_0 to $\mathfrak{usp}(8)$ they will transform in the $\mathbf{10}^{+1}+\overline{\mathbf{10}}^{\,-1}$ with respect to $\mathrm{SO}(2)\times\mathrm{SO}(6)$, $\mathbf{10}^{+1}$ being the same representation as the gravitino mass matrix. In the previous section the itinerary just described from the $N=8$ to the $N=4$ theory was followed backwards: we have reconstructed the 27×27 $\mathfrak{usp}(8)$ matrix T_0 starting from the symmetry $\mathfrak{u}(4)$ of the ungauged $N=4$ action and the fluxes $f_\alpha{}^{IJK}$ defining the gauging.

As far as the fermions are concerned, we note that in the $N=8$ theory, the gravitini in the $\mathbf{8}$ of $\mathrm{USp}(8)$ decompose under $\mathrm{SO}(2)\times\mathrm{SO}(6)_2\subset\mathrm{USp}(8)$ as

$$\mathbf{8}\longrightarrow\mathbf{4}^{+\frac12}+\overline{\mathbf{4}}^{\,-\frac12} \tag{8.15}$$

so that a vector in the $\mathbf{8}$ can be written as

$$V^a=\begin{pmatrix} v_A^{(+\frac12)}\\ v^{A(-\frac12)}\end{pmatrix}\qquad A,B=1,\dots 8\,;\quad a,b=1,\dots 8\,. \tag{8.16}$$

From equation (8.7) we see that the off diagonal generators in the coset $\frac{\mathrm{USp}(8)}{\mathrm{U}(4)}$ belong to the $\mathrm{U}(4)$ representation $\mathbf{10}^{+1}+\overline{\mathbf{10}}^{\,-1}$ among which we find the symplectic invariant

$$\mathbf{C}_{ab}=\begin{pmatrix}\mathbf{0}_{4\times4} & \mathbb{1}_{4\times4}\\ -\mathbb{1}_{4\times4} & \mathbf{0}_{4\times4}\end{pmatrix}. \tag{8.17}$$

The basic quantities which define the fermionic masses and the gradient flows equations of the $N=4$ model (in absence of $D3$-brane couplings) are the symmetric matrices

$$S_{AB}=-\frac{i}{48}\left(\overline{F}^{IJK-}+\overline{C}^{IJK-}\right)(\Gamma_{IJK})_{AB} \tag{8.18}$$

$$N_{AB}=-\frac{1}{48}\left(F^{IJK-}+C^{IJK-}\right)(\Gamma_{IJK})_{AB} \tag{8.19}$$

that belong to the representations $\mathbf{10}^{+1}$, $\mathbf{10}^{-1}$ of $\mathrm{U}(4)$ respectively. Note that they have opposite $\mathrm{U}(1)_R$ weight

$$w[S_{AB}]=-w[N_{AB}]=1\,. \tag{8.20}$$

If we indicate with $\lambda_A^{(\overline{4})}$, $\lambda_{IA}^{(\overline{20})}$ the $\overline{\mathbf{4}}$, $\overline{\mathbf{20}}$ irreducible representations of the $\mathbf{24}$ λ_A^I bulk gaugini, the weights of the left handed gravitini, dilatini and gaugini as given in equation (2.5) give in this case

$$w[\psi_A] = \frac{1}{2}; \quad w[\chi^A] = \frac{3}{2}; \quad w[\lambda_{IA}^{(\overline{20})}] = -\frac{1}{2}; \quad w[\lambda^{A(\overline{4})}] = -\frac{1}{2}; \quad w[\lambda_A^i] = -\frac{1}{2}. \tag{8.21}$$

From equations (5.13)–(5.23) it follows that, by suitable projection on the irreducible representations $\overline{\mathbf{4}}$, $\overline{\mathbf{20}}$, the following mass matrices associated to the various bilinears either depend on the S_{AB} or N_{AB} matrices, according to the following scheme:[6]

$$\begin{aligned}
\chi^{A(\overline{4})}\chi^{B(\overline{4})} &\longrightarrow 0 && (8.22)\\
\chi^{A(\overline{4})}\lambda_{IB}^{(\overline{20})} &\longrightarrow S^{AB} && (8.23)\\
\chi^{A(\overline{4})}\lambda^{B(\overline{4})} &\longrightarrow N_{AB} && (8.24)\\
\lambda_{IA}^{(\overline{20})}\lambda_{JB}^{(\overline{20})} &\longrightarrow S_{AB} && (8.25)\\
\lambda^{A(\overline{4})}\lambda^{B(\overline{4})} &\longrightarrow S_{AB} && (8.26)\\
\lambda^{A(\overline{4})}\lambda_{IB}^{(\overline{20})} &\longrightarrow N^{AB} && (8.27)\\
\psi_A\chi_B^{(4)} &\longrightarrow N^{AB} && (8.28)\\
\psi_A\lambda_B^{(4)} &\longrightarrow S^{AB} && (8.29)\\
\psi_A\lambda^{IB(20)} &\longrightarrow N_{AB} && (8.30)\\
\psi_A\psi_B &\longrightarrow S^{AB} && (8.31)\\
\lambda_A^i\lambda_B^j &\longrightarrow N^{AB} && (8.32)
\end{aligned}$$

All these assignments come from the fact that S_{AB}, N_{AB} are in the $\mathbf{10}^{+1}$ $\mathbf{10}^{-1}$ representations of U(4) and the mass matrices must have grading opposite to the bilinear fermions, since the lagrangian has zero grading. Indeed, from the group theoretical decomposition we find, for each of the listed bilinear fermions

$$\begin{aligned}
\overline{\mathbf{4}}^{\frac{3}{2}} \times \overline{\mathbf{4}}^{\frac{3}{2}} &\not\supset \mathbf{10}^{\pm 1} && (8.33)\\
\overline{\mathbf{4}}^{\frac{3}{2}} \times \overline{\mathbf{20}}^{\,-\frac{1}{2}} &\supset \mathbf{10}^{+1} && (8.34)\\
\overline{\mathbf{4}}^{\frac{3}{2}} \times \overline{\mathbf{4}}^{\,-\frac{1}{2}} &\supset \overline{\mathbf{10}}^{\,+1} && (8.35)\\
\overline{\mathbf{20}}^{\,-\frac{1}{2}} \times \overline{\mathbf{20}}^{\,-\frac{1}{2}} &\supset \mathbf{10}^{-1} + \overline{\mathbf{10}}^{\,-1} && (8.36)\\
\overline{\mathbf{4}}^{\,-\frac{1}{2}} \times \overline{\mathbf{4}}^{\,-\frac{1}{2}} &\supset \overline{\mathbf{10}}^{\,-1} && (8.37)\\
\overline{\mathbf{4}}^{\,-\frac{1}{2}} \times \overline{\mathbf{20}}^{\,-\frac{1}{2}} &\supset \mathbf{10}^{-1} && (8.38)\\
\mathbf{4}^{\frac{1}{2}} \times \mathbf{4}^{-\frac{3}{2}} &\supset \mathbf{10}^{-1} && (8.39)\\
\mathbf{4}^{\frac{1}{2}} \times \mathbf{4}^{\frac{1}{2}} &\supset \mathbf{10}^{+1} && (8.40)\\
\mathbf{4}^{\frac{1}{2}} \times \mathbf{20}^{\frac{1}{2}} &\supset \overline{\mathbf{10}}^{\,+1} && (8.41)\\
\mathbf{4}^{\frac{1}{2}} \times \mathbf{4}^{\frac{1}{2}} &\supset \mathbf{10}^{\,+1} && (8.42)\\
\mathbf{4}^{-\frac{1}{2}} \times \mathbf{4}^{-\frac{1}{2}} &\supset \mathbf{10}^{-1}. && (8.43)
\end{aligned}$$

[6]We remind that (S^{AB}, N^{AB}) have opposite U(1) weights, since they transform in the complex conjugate representation with respect to (S_{AB}, N_{AB}).

The decomposition of the $\overline{\mathbf{20}}^{\,-\frac{1}{2}} \times \overline{\mathbf{20}}^{\,-\frac{1}{2}}$ implies that in principle we have both S_{AB} and N^{AB} appearing in the $\lambda^{(\overline{20})}_{IA}\lambda^{(\overline{20})}_{JB}$ mass term. However an explicit calculation shows that the representation $\mathbf{10}^{-1}$, corresponding to N_{AB} is missing.

The above assignments are consistent with the Scherk-Schwarz truncation of $N = 8$ supergravity [37], where the two matrices Q_{5ab}, P_{5abcd} contain the $\mathbf{10}, \overline{\mathbf{10}}$ of SU(4) of the $N = 4$ theory. More explicitly:

$$
\begin{aligned}
Q_{5ab} &\longrightarrow (S_{AB}, S^{AB}) \\
P_{5abcd} &\longrightarrow (N_{AB}, N^{AB})
\end{aligned}
$$

which is consistent with the fact that, on the vacuum, $P_{5abcd} = 0$ in the Scherk-Schwarz $N = 8$ model and $N_{AB} = 0$ in our $N = 4$ orientifold model.

Let us consider now the decomposition of the dilatino in the $\mathbf{48}$ of USp(8) under U(4). We get:

$$
\chi_{abc} \longrightarrow \chi_{ABC} \oplus \chi^{AB}{}_{C} + h.c. \tag{8.44}
$$

corresponding to

$$
\mathbf{48} \longrightarrow \overline{\mathbf{4}} + \overline{\mathbf{20}} + \mathbf{4} + \mathbf{20}\,. \tag{8.45}
$$

We may then identify the chiral dilatino and gaugino as follows:

$$
\chi^{A} = \epsilon^{ABCD}\chi_{BCD}\,; \qquad \lambda^{I(\overline{20})}_{C} = (\Gamma^{I})_{AB}\chi^{AB}{}_{C}\,. \tag{8.46}
$$

Moreover the decomposition $\mathbf{8} \longrightarrow \mathbf{4}^{+\frac{1}{2}} + \overline{\mathbf{4}}^{\,-\frac{1}{2}}$ identifies $\mathbf{4}^{+\frac{1}{2}}$ with $\lambda^{I(4)}_{A}$ and $\overline{\mathbf{4}}^{\,-\frac{1}{2}}$ with $\lambda^{IA(\overline{4})}$ as they come from the $\mathbf{C}$-trace part or the threefold antisymmetric product $\mathbf{8}\times\mathbf{8}\times\mathbf{8}$.

These results are consistent with the explicit reduction appearing in reference [37]. Indeed the mass term of reference [37] are of the following form[7]

$$
Q_{5ab}\bar{\psi}'^{a}_{\mu}\gamma^{\mu\rho}\psi'^{b}_{\rho} \tag{8.47}
$$
$$
Q_{5ab}\overline{\zeta}'^{a}\zeta'^{b} \tag{8.48}
$$
$$
Q_{5ab}\bar{\psi}'^{a}_{\mu}\gamma^{\mu}\zeta'^{b} \tag{8.49}
$$
$$
Q_{5a}{}^{e}\bar{\chi}'^{abc}\chi'_{ebc} \tag{8.50}
$$
$$
P_{5}^{abcd}\bar{\psi}'_{\mu a}\gamma^{\mu}\gamma^{5}\chi'_{bcd} \tag{8.51}
$$
$$
P_{5}^{abcd}\overline{\zeta}'_{a}\gamma^{5}\chi'_{bcd}\,. \tag{8.52}
$$

The term (8.47) gives rise to the mass term of the gravitino $S^{AB}\bar{\psi}_{A\mu}\gamma^{\mu\nu}\psi_{B\nu}$; the term $N^{AB}\psi_{A\nu}\gamma^{\mu}\chi_{B}$ and the term $Z^{I}_{A}{}^{B}\bar{\psi}_{A\mu}\gamma^{\mu}\lambda^{IB}$ are obtained by reduction of the structures (8.49), (8.51) via the decompositions (8.39), (8.40). The mass term of the bulk gaugini $T^{AB}_{IJ}\bar{\lambda}^{I}_{A}\lambda^{J}_{B}$ is reconstructed from the terms (8.48), (8.50), (8.52) through the decompositions (8.37), (8.36), (8.38). Finally, the mass term $Q^{I}_{A}{}^{B}\bar{\chi}^{A}\lambda^{I}_{B}$ is obtained by reducing equation (8.50), (8.52) via the decomposition (8.48), (8.50).

[7] Note that the terms (8.48), (8.49) do not appear explicitly in the lagrangian of reference [37] but they would appear after diagonalization of the fermionic kinetic terms.

In conclusion we see that our theory can be thought as a truncation of the Scherk-Schwarz $N=8$ supergravity. Once the Goldstino $\bar{\lambda}^{A(\bar{4})}$ is absorbed to give mass to the gravitino $\psi_{A\mu}$ the spin $\frac{1}{2}$ mass matrix is given by the entries $(\chi\chi,\ \chi\lambda^{(\overline{20})},\ \lambda^{(\overline{20})}\lambda^{(\overline{20})})$. Therefore the full spin $\frac{1}{2}$ mass spectrum is the truncation of the Scherk-Schwarz $N=8$ spin $\frac{1}{2}$ to this sectors.

This justifies the results for the mass spectrum given in section 7. Analogous considerations can be done for the scalar sector.

We conclude by arguing that there is a duality between two microscopically different theories:

$$[\text{type-IIB on an orientifold with fluxes}] \leftrightarrow \begin{bmatrix} N=4 \text{ truncation of } N=8 \text{ theories} \\ \text{spontaneously broken à la Scherk-Schwarz} \end{bmatrix}$$

since they are described by the same $N=4$ four dimensional effective field theory.

Finally we consider the fermionic bilinear involving $D3$-brane gaugini λ^i_A. From the structure of the matrices $W_{iA}^{\ \ B}$, $R_{iA}^{\ \ B}$, V^{Ii}_{AB}, equations (5.18), (5.20), (5.23), we notice that they vanish when the $D3$-brane coordinates commute (i.e. the scalars q^I_i are in the Cartan subalgebra of G).

The diagonal mass U^{ij}_{AB} (5.22) has a gravitational part $\delta^{ij}N_{AB}$ which vanishes on the vacuum while the second term is non vanishing for those gaugini which are not in the Cartan subalgebra of G. Indeed, for $G=\mathrm{SU}(N)$, there are exactly $N(N-1)$ ($\frac{1}{2}$ BPS) vector multiplets which become massive when $\mathrm{SU}(N)$ is spontaneously broken to $\mathrm{U}(1)^{N-1}$.

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A. The solution of the Bianchi identities and the supersymmetry transformation laws

In this appendix we describe the geometric approach for the derivation of the $N=4$ supersymmetry transformation laws of the physical fields.

The first step to perform is to extend the physical fields to superfields in $N=4$ superspace: that means that the space-time 1-forms $\omega^{a\,b}$, V^a,ψ^A, ψ_A, $A_{\Lambda\alpha}$, A_i and the space-time zero-forms χ^A, χ_A, λ^I_A, λ^{IA}, λ^i_A, λ^{iA}, L^α, $E_{\Lambda I}$, $B^{\Lambda\Sigma}$, a^Λ_i are promoted to one-superforms and zero-superforms in $N=4$ superspace, respectively.

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As a consequence the superforms must depend on the superspace coordinates $\{x^\mu; \theta^\alpha_A\}$ (where x^μ, $\mu = 1,2,3,4$ are the ordinary space-time coordinates and θ^α_A, $\alpha = 1,2,3,4$, $A = 1,2,3,4$ are anticommuting fermionic coordinates) in such a way that projected on the space-time submanifold (i.e. setting $\theta^\alpha_A = 0, d\theta^\alpha_A = 0$) they correspond to the ordinary physical fields.

A basis of one-forms on the superspace is given by $\{V^a, \psi^\alpha_A\}$, $a = 1,2,3,4$; here V^a are the vierbein, and ψ^α_A are the fermionic vielbein identified with the gravitini fields.

The appropriate definition for the super-curvatures (or super-field strengths)of the superfield p-forms in the $N = 4$ superspace is[8] as follows (we omit for simplicity the sign of wedge product):

$$
\begin{aligned}
R^{ab} &= d\omega^{ab} - \omega^a_c \omega^{cb} && \text{(A.1)}\\
T^a &= \mathcal{D}V^a - i\bar{\psi}_A \gamma^a \psi^A = 0 && \text{(A.2)}\\
F_{\Lambda\alpha} &= dA_{\Lambda\alpha} - \frac{1}{2} L_\alpha E^I_\Lambda (\Gamma_I)^{AB} \bar{\psi}_A \psi_B - \frac{1}{2}\overline{L}_\alpha E^I_\Lambda (\Gamma_I)_{AB} \bar{\psi}^A \psi^B && \text{(A.3)}\\
F_i &= dA_i - \frac{1}{2} L_2 q^I_i (\Gamma_I)^{AB} \bar{\psi}_A \psi_B - \frac{1}{2}\overline{L}_2 q^I_i (\Gamma_I)_{AB} \bar{\psi}^A \psi^B && \text{(A.4)}\\
\rho_A &= \mathcal{D}\psi_A + \frac{1}{2} q \psi_A - 2Q_A^{\ B} \psi_B && \text{(A.5)}\\
\nabla\chi^A &= \mathcal{D}\chi^A - \frac{3}{2} q \chi^A - 2Q^A_{\ B} \chi^B && \text{(A.6)}\\
\nabla\lambda_{IA} &= \mathcal{D}\lambda_{IA} - \frac{1}{2} q \lambda_{IA} - 2Q_A^{\ B} \lambda_{IB} + \omega_2^{IJ} \lambda_{JA} && \text{(A.7)}\\
\nabla\lambda_{iA} &= \mathcal{D}\lambda_{iA} - \frac{1}{2} q \lambda_{iA} - 2Q_A^{\ B} \lambda_{iB} && \text{(A.8)}\\
p &= -i\epsilon_{\alpha\beta} L^\alpha dL^\beta && \text{(A.9)}\\
P^{IJ} &= -\frac{1}{2}(E dE^{-1} + dE^{-1} E)^{IJ} + \frac{1}{2}\left\{E\left[\nabla B - \frac{1}{2}\left(\nabla a\, a^T - a\,\nabla a^T\right)\right]E\right\}^{IJ} && \text{(A.10)}\\
P^{Ii} &= \frac{1}{2} E^I_\Lambda \nabla a^{\Lambda i}\,. && \text{(A.11)}
\end{aligned}
$$

∇ is the covariant derivative with respect to all the connections that act on the field, including the gauge contribution, while $\mathcal{D}$ is the Lorentz covariant derivative acting on a generic vector A^a and a generic spinor θ respectively as follows

$$
\mathcal{D}A^a \equiv dA^a - \omega^{ab} A_b\,; \qquad \mathcal{D}\theta \equiv d\theta - \frac{1}{4}\omega^{ab}\gamma_{ab}\theta\,. \tag{A.12}
$$

The coefficients appearing in front of the U(1) connection q correspond to the different U(1) weights of the fields as shown in equation (2.5).

$Q^A_{\ B}$ is the R-symmetry $SU(4)_1$ connection, that in terms of the gauged $SO(6)_1$ connection ω_1^{IJ} reads as $Q^A_{\ B} = \frac{1}{8}(\Gamma_{IJ})_A^{\ B}\omega_1^{IJ}$ (see appendix D for details).

Equation (A.2) is a superspace constraint imposing the absence of supertorsion, on the $N = 4$ superspace.

[8]Here and in the following by "curvatures" we mean not only two-forms, but also the one-forms defined as covariant differentials of the zero-form superfields

Note that the definition of the "curvatures" has been chosen in such a way that in absence of vector multiplets the equations by setting $R^{ab} = T^a = \rho^A = \rho_A = F^I = 0$, $I = 1, \dots, 6$ give the Maurer-Cartan equations of the $N = 4$ Poincaré superalgebra dual to the $N = 4$ superalgebra of (anti)-commutators, (the 1-forms $\omega^{ab}, V^a, \psi^A, \psi_A, A^I$ are dual to the corresponding generators of the supergroup).

By d-differentiating the supercurvatures definition (A.1)–(A.11), one obtains the Bianchi identities that are their integrability conditions:

$$
\begin{aligned}
\mathcal{R}^{ab} V_b + i\bar{\psi}_A \gamma^a \rho^A + i\bar{\psi}^A \gamma^a \rho_A &= 0 \\
\mathcal{D} R^{ab} &= 0 \\
\nabla F_{\Lambda\alpha} - L_\alpha E^I_\Lambda (\Gamma_I)^{AB} \bar{\psi}_A \rho_B - \overline{L}_\alpha E^I_\Lambda (\Gamma_I)_{AB} \bar{\psi}^A \rho^B + \frac{1}{2} \nabla L_\alpha E^I_\Lambda (\Gamma_I)^{AB} \bar{\psi}_A \psi_B + & \\
+ \frac{1}{2} \nabla \overline{L}_\alpha E^I_\Lambda (\Gamma_I)_{AB} \bar{\psi}^A \psi^B + \frac{1}{2} L_\alpha \nabla E^I_\Lambda (\Gamma_I)^{AB} \bar{\psi}_A \psi_B + \frac{1}{2} \overline{L}_\alpha \nabla E^I_\Lambda (\Gamma_I)_{AB} \bar{\psi}^A \psi^B &= 0 \\
\nabla F_i - L_2 q^I_i (\Gamma_I)^{AB} \bar{\psi}_A \rho_B - \overline{L}_2 q^I_i (\Gamma_I)_{AB} \bar{\psi}^A \rho^B + \frac{1}{2} \nabla L_2 q^I_i (\Gamma_I)^{AB} \bar{\psi}_A \psi_B + & \\
+ \frac{1}{2} \nabla \overline{L}_2 q^I_i (\Gamma_I)_{AB} \bar{\psi}^A \psi^B + \frac{1}{2} L_2 \nabla q^I_i (\Gamma_I)^{AB} \bar{\psi}_A \psi_B + \frac{1}{2} \overline{L}_2 \nabla q^I_i (\Gamma_I)_{AB} \bar{\psi}^A \psi^B &= 0 \\
\nabla \rho_A + \frac{1}{4} \mathcal{R}^{ab} \gamma_{ab} \psi_A - \frac{1}{2} R \psi_A + 2 R_A^{\ B} \psi_B &= 0 \\
\nabla^2 \chi^A + \frac{1}{4} \mathcal{R}^{ab} \gamma_{ab} \chi^A + \frac{3}{2} R \chi^A + 2 R^A_{\ B} \chi^B &= 0 \\
\nabla^2 \lambda_{IA} + \frac{1}{4} \mathcal{R}^{ab} \gamma_{ab} \lambda_{IA} + \frac{1}{2} R \lambda_{IA} + 2 R_A^{\ B} \lambda_{IB} - R_{2IJ} \lambda^J_A &= 0 \\
\nabla^2 \lambda_{iA} + \frac{1}{4} \mathcal{R}^{ab} \gamma_{ab} \lambda_{iA} + \frac{1}{2} R \lambda_{iA} + 2 R_A^{\ B} \lambda_{iB} &= 0 \\
\nabla p &= 0 \\
\nabla P^{IJ} + \frac{1}{2} E^I_\Lambda (f^{\Lambda\Sigma\Gamma\alpha} F_{\Gamma\alpha} + c^{ijk} a^\Lambda_i a^\Sigma_k F_j) E^J_\Sigma &= 0 \\
\nabla P^{Ii} + \frac{1}{2} a^\Lambda_k E^I_\Lambda c^{ijk} F_j &= 0 \,.
\end{aligned}
\tag{A.13}
$$

Here $R_A^{\ B} = \frac{1}{8} (\Gamma_{IJ})_A^{\ B} R_1^{IJ}$ is the gauged SU(4) curvature with

$$
R_1^{IJ} = d\omega_1^{IJ} + \omega^{IK} \wedge \omega_K^{\ J} - \frac{1}{2} E^I_\Lambda f^{\Lambda\Sigma\Gamma\alpha} F_{\Gamma\alpha} E^J_\Sigma \tag{A.14}
$$

and $R = dq$ is the U(1) curvature.

The solution can be obtained as follows: first of all one requires that the expansion of the curvatures along the intrinsic p-forms basis in superspace namely: $V^a, V^a \wedge V^b, \psi, \psi \wedge V^b, \psi \wedge \psi$, is given in terms only of the physical fields (rheonomy). This insures that no new degree of freedom is introduced in the theory.

Secondly one writes down such expansion in a form which is compatible with all the symmetries of the theory, that is: covariance under U(1) and $\mathrm{SO}_d(6) \otimes \mathrm{SO}(n)$, Lorentz transformations and reparametrization of the scalar manifold. Besides it is very useful to take into account the invariance under the following rigid rescalings of the fields (and their corresponding curvatures):

$$
(\omega^{ab}, A_{\Lambda\alpha}, E^I_\Lambda, B^{\Lambda\Sigma}, a^\Lambda_i) \to (\omega^{ab}, A_{\Lambda\alpha}, E^I_\Lambda, B^{\Lambda\Sigma}, a^\Lambda_i) \tag{A.15}
$$

$$V^a \to \ell V^a \tag{A.16}$$
$$(\psi^A, \psi_A) \to \ell^{\frac{1}{2}}(\psi^A, \psi_A) \tag{A.17}$$
$$(\lambda_{iA}, \lambda^I_A, \chi^A) \to \ell^{-\frac{1}{2}}(\lambda_{iA}, \lambda^I_A, \chi^A) \tag{A.18}$$

Indeed the first three rescalings and the corresponding ones for the curvatures leave invariant the definitions of the curvatures and the Bianchi identities. The last one follows from the fact that in the solution for the σ- model vielbeins p, P^{IJ}, P^{Ii} the spin $\frac{1}{2}$ fermions must appear contracted with the gravitino 1-form.

Performing all the steps one finds the final parametrizations of the superspace curvatures, namely:

$$\begin{aligned}
F_{\Lambda\alpha} &= \mathcal{F}^{ab}_{\Lambda\alpha} V_a V_b + i\Big(\overline{L}_\alpha E^I_\Lambda (\Gamma_I)^{AB} \bar{\chi}_A \gamma_a \psi_B + L_\alpha E^I_\Lambda (\Gamma_I)_{AB} \bar{\chi}^A \gamma_a \psi^B + \\
&\qquad + L_\alpha E^I_\Lambda \bar{\lambda}^A_I \gamma_a \psi_A + \overline{L}_\alpha E^I_\Lambda \bar{\lambda}_{IA} \gamma_a \psi^A\Big) V^a & \text{(A.19)}\\
F_i &= \mathcal{F}^{ab}_i V_a V_b + i\Big(\overline{L}_2 q^I_i (\Gamma_I)^{AB} \bar{\chi}_A \gamma_a \psi_B + L_2 q^I_i (\Gamma_I)_{AB} \bar{\chi}^A \gamma_a \psi^B + L_2 q^I_i \bar{\lambda}^A_I \gamma_a \psi_A + \\
&\qquad + \overline{L}_2 q^I_i \bar{\lambda}_{IA} \gamma_a \psi^A + 2L_2 \bar{\lambda}^A_i \gamma_a \psi_A + 2\overline{L}_2 \bar{\lambda}_{iA} \gamma_a \psi^A\Big) V^a & \text{(A.20)}\\
\rho_A &= \rho_{Aab} V^a V^b - \overline{L}^\alpha (E^{-1})^\Lambda_I (\Gamma^I)_{AB} \mathcal{F}^{-ab}_{\Lambda\alpha} \gamma_b \psi^B V_a + \\
&\quad + \epsilon_{ABCD} \chi^B \bar{\psi}^C \psi^D + S_{AB} \gamma_a \psi^B V^a & \text{(A.21)}\\
\nabla\chi^A &= \nabla\chi^A_a V^a + \frac{i}{2} \overline{p}_a \gamma^a \psi^A + \frac{i}{4} \overline{L}^\alpha (E^{-1})^\Lambda_I (\Gamma^I)^{AB} \mathcal{F}^{-ab}_{\Lambda\alpha} \gamma_{ab} \psi_B + N^{AB} \psi_B & \text{(A.22)}\\
\nabla\lambda_{IA} &= \nabla\lambda_{IAa} V^a + \frac{i}{2} (\Gamma_J)_{AB} P^{JI}_a \gamma^a \psi^B - \frac{i}{2} L^\alpha (E^{-1})^\Lambda_I \mathcal{F}^{-ab}_{\Lambda\alpha} \gamma_{ab} \psi_A + Z_{IA}{}^B \psi_B & \text{(A.23)}\\
\nabla\lambda_{iA} &= \nabla\lambda_{iAa} V^a + \frac{i}{2} (\Gamma_I)_{AB} P^{iI}_a \gamma^a \psi^B - \frac{1}{4\overline{L}_2} q^I_i (E^{-1})^\Lambda_I \mathcal{F}^{-ab}_{\Lambda 2} \gamma_{ab} \psi_A + \\
&\quad + \frac{1}{4\overline{L}_2} \mathcal{F}^{-ab}_i \gamma_{ab} \psi_A + W_{iA}{}^B \psi_B & \text{(A.24)}\\
p &= p_a V^a + 2\bar{\chi}_A \psi^A & \text{(A.25)}\\
P^{IJ} &= P^{IJ}_a V^a + (\Gamma^I)^{AB} \bar{\lambda}^J_A \psi_B + (\Gamma^I)_{AB} \bar{\lambda}^{JA} \psi^B & \text{(A.26)}\\
P^{Ii} &= P^{Ii}_a V^a + (\Gamma^I)^{AB} \bar{\lambda}^i_A \psi_B + (\Gamma^I)_{AB} \bar{\lambda}^{iA} \psi^B\,. & \text{(A.27)}
\end{aligned}$$

The previous parametrizations are given up to three fermions terms, except equation (A.21) where the term with two gravitini has been computed; in fact this term is in principle involved in the computation of the gravitino shift but by explicit computation its contribution vanishes

As promised the solution for the curvatures is given as an expansion along the 2-form basis $(V \wedge V\,,\, V \wedge \psi\,,\, \psi \wedge \psi)$ or the 1-form basis $(V\,,\,\psi)$ with coefficients given in terms of the physical fields.

It is important to stress that the components of the field strengths along the bosonic vielbeins are not the space-time field strengths since $V^a = V^a_\mu dx^\mu + V^a_\alpha d\theta^\alpha$ where $(V^a_\mu\,,\, V^a_\alpha)$ is a submatrix of the super-vielbein matrix $E^I \equiv (V^a\,,\,\psi)$. The physical field strengths are given by the expansion of the forms along the dx^μ-differentials and by restricting the superfields to space-time ($\theta = 0$ component). For example, from the parametrization (A.19), expanding along the dx^μ-basis one finds:

$$
\begin{aligned}
F^\Lambda_{\mu\nu} &= F^\Lambda_{ab} V^a_{[\mu} V^b_{\nu]} + \mathrm{i}\overline{L}_\alpha E^I_\Lambda (\Gamma_I)^{AB} \bar\chi_A \gamma_{[\mu} \psi_{\nu]B} + \mathrm{i} L_\alpha E^I_\Lambda (\Gamma_I)_{AB} \bar\chi^A \gamma_{[\mu} \psi^B_{\nu]} \\
&\quad + i L_\alpha E^I_\Lambda \bar\lambda^A_I \gamma_{[\mu} \psi_{\nu]A} + \mathrm{i}\overline{L}_\alpha E^I_\Lambda \bar\lambda_{IA} \gamma_{[\mu} \psi^A_{\nu]}
\end{aligned}
\tag{A.28}
$$

where $F^\Lambda_{\mu\nu}$ is defined by the expansion of eq. (A.3) along the dx^μ-differentials. When all the superfields are restricted to space-time we may treat the V^a_μ vielbein as the usual 4-dimensional invertible matrix converting intrinsic indices in coordinate indices and we see that the physical field-strength $\mathcal{F}_{\Lambda\alpha\mu\nu}$ differs from $F_{\Lambda\alpha\, ab} V^a_{[\mu} V^b_{\nu]} \equiv \tilde F_{\Lambda\alpha\mu\nu}$ by a set of spinor currents ($\tilde F_{\Lambda\alpha\mu\nu}$ is referred to as the supercovariant field-strength).

Analougous considerations hold for the other field-strengths components along the bosonic vielbeins.

Note that the solution of the Bianchi identities also implies a set of differential constraints on the components along the bosonic vielbeins which are to be identified, when the fields are restricted to space-time only, with the equations of motion of the theory. Indeed the analysis of the Bianchi identities for the fermion fields give such equations (in the sector containing the 2-form basis $\bar\psi_A \gamma^a \psi^A$). Further the superspace derivative along the ψ_A $\left(\psi^A\right)$ directions, which amounts to a supersymmetry transformation, yields the equations of motion of the bosonic fields. Indeed the closure of the Bianchi identities is equivalent to the closure of the $N=4$ supersymmetry algebra on the physical fields and we know that in general such closure implies the equations of motion .

The determination of the superspace curvatures enables us to write down the $N=4$ SUSY transformation laws. Indeed we recall that from the superspace point of view a supersymmetry transformation is a Lie derivative along the tangent vector:

$$
\epsilon = \bar\epsilon^A \vec D_A + \bar\epsilon_A \vec D^A \tag{A.29}
$$

where the basis tangent vectors $\vec D_A$, $\vec D^A$ are dual to the gravitino 1-forms:

$$
\vec D_A \left(\psi^B\right) = \vec D^A \left(\psi_B\right) = \mathbf{1} \tag{A.30}
$$

and $\mathbf{1}$ is the unit in spinor space.

Denoting by μ^I and R^I the set of one-forms $\left(V^a,\, \psi_A,\, \psi^A,\, A_{\Lambda\alpha}, A_i\right)$ and of two-forms $\left(T^a = 0,\, \rho_A,\, \rho^A,\, F_{\Lambda\alpha}, F_i\right)$ respectively, one has:

$$
\ell \mu^I = \left(i_\epsilon d + d i_\epsilon\right) \mu^I \equiv \left(D\epsilon\right)^I + i_\epsilon R^I \tag{A.31}
$$

where D is the derivative covariant with respect to the $N=4$ Poincaré superalgebra and i_ϵ is the contraction operator along the tangent vector ϵ.

In our case:

$$
\begin{aligned}
(D\epsilon)^a &= -\mathrm{i}\left(\bar\psi_A \gamma^a \epsilon^A + \bar\psi^A \gamma^a \epsilon_A\right) &\text{(A.32)}\\
(D\epsilon)^\alpha &= \nabla \epsilon^\alpha &\text{(A.33)}\\
(D\epsilon)_{\Lambda\alpha} &= (D\epsilon)_i = 0 &\text{(A.34)}
\end{aligned}
$$

(here α is a spinor index).

For the 0-forms which we denote shortly as ν^I we have the simpler result:

$$\ell_\epsilon = i_\epsilon d\nu^I = i_\epsilon \left(\nabla \nu^I - \text{connection terms}\right) \tag{A.35}$$

Using the parametrizations given for R^I and $\nabla \nu^I$ and identifying δ_ϵ with the restriction of ℓ_ϵ to space-time it is immediate to find the $N = 4$ susy laws for all the fields. The explicit formulae are given by the equations (5.24).

B. Derivation of the space time lagrangian from the geometric approach

We have seen how to reconstruct the $N = 4$ susy transformation laws of the physical fields from the solution of the Bianchi identities in superspace.

In principle, since the Bianchi identities imply the equations of motion, the lagrangian could also be completely determined. However this would be a cumbersome procedure.

In this appendix we give a short account of the construction of the lagrangian on space-time from a geometrical lagrangian in superspace. Note that while the solution of the Bianchi identities is completely equivalent to the ordinary "Superspace approach" (apart from notations and a different emphasis on the group-theoretical structure),the geometric approach for the construction of the lagrangian is completely different from the usual superspace approach via integration in superspace.

In the geometric (rheonomic) approach the superspace action is a 4-form in superspace integrated on a 4-dimensional (bosonic) hypersurface $\mathcal{M}^4$ locally embedded in superspace

$$\mathcal{A} = \int_{\mathcal{M}^4 \subset \mathcal{SM}} \mathcal{L} \tag{B.1}$$

where $\mathcal{SM}$ is the superspace manifold. Provided we do not introduce the Hodge duality operator in the construction of $\mathcal{L}$ the equations of motions derived from the generalized variational principle $\delta\mathcal{A} = 0$ are 3-form or 4-form equations independent from the particular hypersurface $\mathcal{M}^4$ on which we integrate and they are therefore valid in all superspace. (Indeed in the variational principle we have also to vary the hypersurface which can always compensated by a diffeomorphism of the fields if the lagrangian is written olnly in terms of differential forms).

These superspace equations of motion can be analyzed along the 3-form basis. The components of the equations obtained along bosonic vielbeins give the differential equations for the fields which, identifying $\mathcal{M}^4$ with space-time, are the ordinary equations of motion of the theory. The components of the same equations along 3-forms containing at least one gravitino ("outer components") give instead algebraic relations which identify the components of the various "supercurvatures" in the outer directions in terms of the physical fields along the bosonic vierbeins (rhenomy principle).

Actually if we have already solved the Bianchi identities this requirement is equivalent to identify the outer components of the curvatures obtained from the variational principle with those obtained from the Bianchi identities.

There are simple rules which can be used in order to write down the most general lagrangian compatible with this requirement.

The implementation of these rules is described in detail in the literature [59] to which we refer the interested reader. Actually one writes down the most general 4-form as a sum of terms with indeterminate coefficients in such a way that $\mathcal{L}$ be a scalar with respect to all the symmetry transformations of the theory (Lorentz invariance,U(1), $\mathrm{SO}(6)_d \otimes \mathrm{SO}(n)$ invariance, invariance under the rescaling (A.18). Varying the action and comparing the outer equations of motion with the actual solution of the Bianchi identities one then fixes all the undetermined coefficients.

Let us perform the steps previously indicated. The most general lagrangian has the following form: (we will determine the complete lagrangian up to four fermion terms):

$$
\begin{aligned}
\mathcal{L}_{\text{kinetic}} =\;& \mathcal{R}^{ab} V^c V^d \epsilon_{abcd} + a_1 \left(\bar{\psi}_A \gamma_a \rho^A V^a - \bar{\psi}^A \gamma_a \rho_A V^a\right) + \\
&+ [a_2 \left(\bar{\chi}^A \gamma^a \nabla \chi_A + \bar{\chi}_A \gamma^a \nabla \chi^A\right) + a_3 \left(\bar{\lambda}^{IA} \gamma^a \nabla \lambda_{IA} + \bar{\lambda}_{IA} \gamma^a \nabla \lambda^{IA}\right) + \\
&+ a_6 (\bar{\lambda}^{iA} \gamma^a \nabla \lambda_{iA} + \bar{\lambda}_{iA} \gamma^a \nabla \lambda^{iA}) V^b V^c V^d \epsilon_{abcd} + \\
&+ a_4 \left[\mathbf{p}^a (\overline{p} - 2\bar{\chi}^A \psi_A) + \overline{\mathbf{p}}^a (p - 2\bar{\chi}_A \psi^A) - \frac{1}{4} \overline{\mathbf{p}}_f \mathbf{p}^f V^a\right] V^b V^c V^d \epsilon_{abcd} + \\
&+ a_5 \left[\mathbf{P}^{IJ}_a (P^{IJ} - (\Gamma^I)^{AB} \bar{\lambda}^J_A \psi_B - (\Gamma^I)_{AB} \bar{\lambda}^{JA} \psi^B) - \frac{1}{8} \mathbf{P}^{IJ}_f \mathbf{P}^{IJ\,f} V^a\right] V^b V^c V^d \epsilon_{abcd} + \\
&+ a_7 \left[\mathbf{P}^{Ii}_a (P^{Ii} - (\Gamma^I)^{AB} \bar{\lambda}^i_A \psi_B - (\Gamma^I)_{AB} \bar{\lambda}^{iA} \psi^B) - \frac{1}{8} \mathbf{P}^{Ii}_f \mathbf{P}^{Ii\,f} V^a\right] V^b V^c V^d \epsilon_{abcd} + \\
&+ a \left[\mathcal{N}^{\Lambda\alpha\Sigma\beta} \mathbf{F}^{+ab}_{\Lambda\alpha} + \overline{\mathcal{N}}^{\Lambda\alpha\Sigma\beta} \mathbf{F}^{-ab}_{\Lambda\alpha} + \mathcal{N}^{i\Sigma\beta} \mathbf{F}^{+ab}_i + \overline{\mathcal{N}}^{i\Sigma\beta} \mathbf{F}^{-ab}_i\right] \times \\
&\times \Big[F_{\Sigma\beta} - i \Big(\overline{L}_\beta E^I_\Sigma (\Gamma_I)^{AB} \bar{\chi}_A \gamma^d \psi_B + L_\beta E^I_\Sigma (\Gamma_I)_{AB} \bar{\chi}^A \gamma^d \psi^B + \\
&\qquad + L_\beta E^I_\Sigma \bar{\lambda}^A_I \gamma^d \psi_A + \overline{L}_\beta E^I_\Sigma \bar{\lambda}_{IA} \gamma^d \psi^A\Big) V_d\Big] V_a V_b + \\
&+ a \left[\mathcal{N}^{\Lambda\alpha j} \mathbf{F}^{+ab}_{\Lambda\alpha} + \overline{\mathcal{N}}^{\Lambda\alpha j} \mathbf{F}^{-ab}_{\Lambda\alpha} + \mathcal{N}^{ij} \mathbf{F}^{+ab}_i + \overline{\mathcal{N}}^{ij} \mathbf{F}^{-ab}_i\right] \times \\
&\times \Big[F_j - i \Big(\overline{L}_2 q^I_j (\Gamma_I)^{AB} \bar{\chi}_A \gamma^d \psi_B + L_2 q^I_j (\Gamma_I)_{AB} \bar{\chi}^A \gamma^d \psi^B + \\
&\qquad + L_2 q^I_j \bar{\lambda}^A_I \gamma^d \psi_A + \overline{L}_2 q^I_j \bar{\lambda}_{IA} \gamma^d \psi^A + L_2 \bar{\lambda}^A_j \gamma^d \psi_A + \overline{L}_2 \bar{\lambda}_{jA} \gamma^d \psi^A\Big) V_d\Big] \times \\
&\times V_a V_b - \frac{i}{24} a \left(\mathcal{N}^{\Lambda\alpha\Sigma\beta} \mathbf{F}^{+fg}_{\Lambda\alpha} \mathbf{F}^{+}_{fg\Sigma\beta} - \overline{\mathcal{N}}^{\Lambda\alpha\Sigma\beta} \mathbf{F}^{-fg}_{\Lambda\alpha} \mathbf{F}^{-}_{fg\Sigma\beta}\right) V^a V^b V^c V^d \epsilon_{abcd} + \\
&- \frac{i}{24} a \left(\mathcal{N}^{i\Sigma\beta} \mathbf{F}^{+lm}_i \mathbf{F}^{+}_{fg\Sigma\beta} - \overline{\mathcal{N}}^{i\Sigma\beta} \mathbf{F}^{-fg}_i \mathbf{F}^{-}_{fg\Sigma\beta}\right) V^a V^b V^c V^d \epsilon_{abcd} + \\
&- \frac{i}{24} a \left(\mathcal{N}^{ij} \mathbf{F}^{+fg}_i \mathbf{F}^{+}_{fg\,i} - \overline{\mathcal{N}}^{ij} \mathbf{F}^{-fg}_i \mathbf{F}^{-}_{fg\,j}\right) V^a V^b V^c V^d \epsilon_{abcd} +
\end{aligned}
\tag{B.2}
$$

$$
\begin{aligned}
\mathcal{L}_{\text{Pauli}} =\;& b_1 \left[p \bar{\chi}^A \gamma_{ab} \psi_A - \overline{p} \bar{\chi}_A \gamma_{ab} \psi^A\right] V^a V^b + \\
&+ b_2 P^{IJ} \left[(\Gamma^I)^{AB} \bar{\lambda}^J_A \gamma_{ab} \psi_B - (\Gamma^I)_{AB} \bar{\lambda}^{JA} \gamma_{ab} \psi^B\right] V^a V^b + \\
&+ b_3 P^{Ii} \left[(\Gamma^I)^{AB} \bar{\lambda}^i_A \gamma_{ab} \psi_B - (\Gamma^I)_{AB} \bar{\lambda}^{iA} \gamma_{ab} \psi^B\right] V^a V^b + \\
&+ F_{\Lambda\alpha} \Big[c_1 \Big(\overline{\mathcal{N}}^{\Lambda\alpha\Sigma\beta} L_\beta E^I_\Sigma (\Gamma_I)^{AB} \bar{\psi}_A \psi_B + \mathcal{N}^{\Lambda\alpha\Sigma\beta} \overline{L}_\beta E^I_\Sigma (\Gamma_I)_{AB} \bar{\psi}^A \psi^B\Big) + \\
&\quad + c_2 \left(\mathcal{N}^{\Lambda\alpha\Sigma\beta} \overline{L}_\beta E^I_\Sigma (\Gamma_I)^{AB} \bar{\chi}_A \gamma_a \psi_B + \overline{\mathcal{N}}^{\Lambda\alpha\Sigma\beta} L_\beta E^I_\Sigma (\Gamma_I)_{AB} \bar{\chi}^A \gamma_a \psi^B\right) V^a + \\
&\quad + c_3 (\mathcal{N}^{\Lambda\alpha\Sigma\beta} L_\beta E^I_\Sigma \bar{\lambda}^A_I \gamma_a \psi_A + \overline{\mathcal{N}}^{\Lambda\alpha\Sigma\beta} \overline{L}_\beta \bar{\lambda}_{IA} \gamma_a \psi^A) V^a\Big] +
\end{aligned}
$$

$$
\begin{aligned}
&+F_{\Lambda\alpha}\Big[c_4\left(\overline{\mathcal{N}}^{\Lambda\alpha\, i}L_2 q^I_i(\Gamma_I)^{AB}\bar\psi_A\psi_B+\mathcal{N}^{\Lambda\alpha\, i}\overline{L}_2 q^I_i(\Gamma_I)_{AB}\bar\psi^A\psi^B\right)+\\
&\qquad +c_5\Big(\mathcal{N}^{\Lambda\alpha\, i}\overline{L}_2 q^I_i(\Gamma_I)^{AB}\bar\chi_A\gamma_a\psi_B+\overline{\mathcal{N}}^{\Lambda\alpha\, i}L_2 q^I_i(\Gamma_I)_{AB}\bar\chi^A\gamma_a\psi^B\Big)V^a+\\
&\qquad +c_6\left(\mathcal{N}^{\Lambda\alpha\, i}L_2 q^I_i\bar\lambda^A_I\gamma_a\psi_A+\overline{\mathcal{N}}^{\Lambda\alpha\, i}\overline{L}_2 q^I_i\bar\lambda_{IA}\gamma_a\psi^A\right)V^a+\\
&\qquad +c_7\left(\mathcal{N}^{\Lambda\alpha\, i}L_2\bar\lambda^A_i\gamma_a\psi_A+\overline{\mathcal{N}}^{\Lambda\alpha\, i}\overline{L}_2\bar\lambda_{iA}\gamma_a\psi^A\right)V^a\Big]+\\
&+F_i\Big[c_1\Big(\overline{\mathcal{N}}^{i\Sigma\beta}L_\beta E^I_\Sigma(\Gamma_I)^{AB}\bar\psi_A\psi_B+\mathcal{N}^{i\Sigma\beta}\overline{L}_\beta E^I_\Sigma(\Gamma_I)_{AB}\bar\psi^A\psi^B\Big)+\\
&\qquad +c_2\Big(\mathcal{N}^{i\Sigma\beta}\overline{L}_\beta E^I_\Sigma(\Gamma_I)^{AB}\bar\chi_A\gamma_a\psi_B+\overline{\mathcal{N}}^{i\Sigma\beta}L_\beta E^I_\Sigma(\Gamma_I)_{AB}\bar\chi^A\gamma_a\psi^B\Big)V^a+\\
&\qquad +c_3\Big(\mathcal{N}^{i\Sigma\beta}L_\beta E^I_\Sigma\bar\lambda^A_I\gamma_a\psi_A+\overline{\mathcal{N}}^{i\Sigma\beta}\overline{L}_\beta\bar\lambda_{IA}\gamma_a\psi^A\Big)V^a\Big]+\\
&+F_i\Big[c_4\Big(\overline{\mathcal{N}}^{ij}L_2 q^I_j(\Gamma_I)^{AB}\bar\psi_A\psi_B+\mathcal{N}^{ij}\overline{L}_2 q^I_j(\Gamma_I)_{AB}\bar\psi^A\psi^B\Big)+\\
&+c_5\Big(\mathcal{N}^{ij}\overline{L}_2 q^I_j(\Gamma_I)^{AB}\bar\chi_A\gamma_a\psi_B+\overline{\mathcal{N}}^{ij}L_2 q^I_j(\Gamma_I)_{AB}\bar\chi^A\gamma_a\psi^B\Big)V^a+\\
&+c_6\Big(\mathcal{N}^{ij}L_2 q^I_j\bar\lambda^A_I\gamma_a\psi_A+\overline{\mathcal{N}}^{ij}\overline{L}_2 q^I_j\bar\lambda_{IA}\gamma_a\psi^A\Big)V^a+\\
&+c_7\Big(\mathcal{N}^{ij}L_2\bar\lambda^A_j\gamma_a\psi_A+\overline{\mathcal{N}}^{ij}\overline{L}_2\bar\lambda_{jA}\gamma_a\psi^A\Big)V^a\Big]+\text{more terms} \qquad\qquad (B.3)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{L}_{\rm gauge}=\;& g_1\Big(\bar\psi_A\gamma_{ab}\psi_B S^{AB}-\bar\psi^A\gamma_{ab}\psi^B S_{AB}\Big)V^aV^b+\\
& g_2\Big(\bar\psi_A\gamma^a\chi_B N^{AB}+\bar\psi^A\gamma^a\chi^B N_{AB}\Big)V^bV^cV^d\epsilon_{abcd}+\\
&+g_3\Big(\bar\psi_A\gamma^a\lambda^{IB}Z_{IB}^{\ \ A}+\bar\psi^A\gamma^a\lambda^I_B Z_I^{B}{}_{A}\Big)V^bV^cV^d\epsilon_{abcd}+\\
&+g_4\Big(\bar\psi_A\gamma^a\lambda^{iB}W_{iB}^{\ \ A}+\bar\psi^A\gamma^a\lambda^i_B\ W_i^{B}{}_{A}\Big)V^bV^cV^d\epsilon_{abcd}+\\
&+\Big(\bar\lambda^A_I\chi_B Q^I_A{}^B+\bar\lambda_{IA}\chi^B Q^{IA}{}_B+\bar\lambda^A_i\chi_B M^i_A{}^B+\bar\lambda_{iA}\chi^B M^{iA}{}_B\Big)V^aV^bV^cV^d\epsilon_{abcd}+\\
&+\Big(\bar\lambda_{IA}\lambda_{JB}T^{IJAB}+\bar\lambda^{IA}\lambda^{JB}T_{IJAB}+\bar\lambda_{iA}\lambda_{jB}U^{ijAB}+\\
&\quad+\bar\lambda^{iA}\lambda^{jB}U_{ijAB}\Big)V^aV^bV^cV^d\epsilon_{abcd}+ \qquad\qquad (B.4)\\
&-\frac{1}{24}\Big(-12S_{AC}\overline{S}^{CA}+4\,N_{AC}\overline{N}^{CA}+2Z^{IC}{}_AZ^I_C{}^A+4W^{iC}{}_AW^i_C{}^A\Big)V^aV^bV^cV^d\epsilon_{abcd}\\
\mathcal{L}_{\rm torsion}=\;& T_aV^aV^b\left(t_1\bar\chi^A\gamma_b\chi_a+t_2\bar\lambda_{IA}\gamma_b\lambda^{IA}+t_3\bar\lambda_{iA}\gamma_b\lambda^{iA}\right). \qquad\qquad (B.5)
\end{aligned}
$$

Note that in equation (B.3) the statement "+ more terms" means Pauli terms containing currents made out spin $\frac{1}{2}$ bilinears which can not be computed in this geometric approach without knowledge of the four fermion couplings. However these terms have been included in the space-time lagrangian given in section 5 by imposing the invariance of the space-time lagrangian under supersymmetry transformations.

The introduction of the auxiliary 0-forms $\mathbf{p}^a, \mathbf{P}^{IJ}_a, \mathbf{F}^{\pm ab}_{\Lambda\alpha}, \mathbf{F}^{\pm ab}_i$ is a trick which avoids the use of the Hodge operator for the construction of the kinetic terms for the vectors and scalar fields which otherwise would spoil the validity of the 3-form equations of motion in all superspace; indeed the equation of of motion of these auxiliary 0-forms identifies them with the components of the physical field-strengths $p^a, P^{IJ}_a, F^{\pm ab}_{\Lambda\alpha}, F^{\pm ab}_i$ along the bosonic vielbeins V^a thus reconstructing the usual kinetic terms on space-time.

The $\mathcal{L}_{torsion}$-term has been constructed in such a way as to give $T^a = 0$.

Performing the variation of all the fields one fixes all the undetermined coefficients, namely:

$$
\begin{array}{llllll}
a_1 = 4\,; & a_2 = -\frac{4}{3}i\,; & a_3 = -\frac{2}{3}i\,; & a_4 = \frac{2}{3}\,; & a_5 = \frac{2}{3}\,; & a_6 = -\frac{4}{3}i\,; \\
a_7 = \frac{4}{3}\,; & c_1 = -2\,; & c_2 = -4i\,; & c_3 = -4i\,; & c_4 = -2\,; & c_5 = -4i; \\
c_6 = -4i\,; & c_7 = -8i\,; & b_1 = 4i\,; & b_2 = 2i\,; & b_3 = 4i\,; & \\
g_1 = -4\,; & g_2 = -\frac{8}{3}i\,; & g_3 = -\frac{4}{3}i\,; & g_4 = -\frac{8}{3}i\,; & & \\
t_1 = -4\,; & t_2 = -2\,; & t_3 = -4\,; & a = -4\,. & &
\end{array}
\tag{B.6}
$$

In order to obtain the space-time lagrangian the last step to perform is the restriction of the 4-form lagrangian from superspace to space-time. Namely we restrict all the terms to the $\theta = 0\,,\, d\theta = 0$ hypersurface $\mathcal{M}^4$. In practice one first goes to the second order formalism by identifying the auxiliary 0-form fields as explained before. Then one expands all the forms along the dx^μ differentials and restricts the superfields to their lowest ($\theta = 0$) component. Finally the coefficients of:

$$
dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma \;=\; \frac{\epsilon^{\mu\nu\rho\sigma}}{\sqrt{g}}\left(\sqrt{g}d^4x\right) \tag{B.7}
$$

give the lagrangian density written in section 5. The overall normalization of the space-time action has been chosen such as to be the standard one for the Einstein term. (To conform to the usual definition of the Riemann tensor $R^{ab}{}_{cd}$ we have set $R^{ab} = -\frac{1}{2}R^{ab}{}_{cd}V^cV^d$).

C. The moduli of T^6 in real and complex coordinates

Appendix we give a more detailed discussion of the extrema of the potential using a complex basis for the $\mathrm{GL}(6,\mathbb{R})$ indices for the moduli of the T^6 torus.

Let us consider the basis vectors $\{e_\Lambda\}$, $(\Lambda = 1\ldots 6)$ of the fundamental representation of $\mathrm{GL}(6,\mathbb{R})$. We introduce a complex basis $\{E_i, \overline{E}_i\}$ with $i = 1,\,2,\,3$ or, to avoid confusion on indices, $i = x,\,y,\,z$ in the following way:

$$
e_1 + ie_4 = E_x\,; \qquad e_2 + ie_5 = E_y\,; \qquad e_3 + ie_6 = E_z \tag{C.1}
$$

$$
e_1 - ie_4 = \overline{E}_x\,; \qquad e_2 - ie_5 = \overline{E}_y\,; \qquad e_3 - ie_6 = \overline{E}_z\,. \tag{C.2}
$$

The axion fields and the (inverse) metric of T^6 can then be written using (anti)holomorphic indices $i,\,j,\,\bar{\imath},\,\bar{\jmath}$ as follows:

$$
B^{\Lambda\Sigma} \longrightarrow B^{ij},\, B^{i\bar{\jmath}},\, B^{\bar{\imath}j},\, B^{\bar{\imath}\bar{\jmath}} \tag{C.3}
$$

$$
g^{\Lambda\Sigma} \longrightarrow g^{ij},\, g^{i\bar{\jmath}},\, g^{\bar{\imath}j},\, g^{\bar{\imath}\bar{\jmath}}\,. \tag{C.4}
$$

In particular, the fluxes $f_1^{\Lambda\Sigma\Gamma} \equiv f^{\Lambda\Sigma\Gamma}$ are given by

$$
f^{xyz} = \frac{1}{8}\left\{f^{123} - f^{156} + f^{246} - f^{345} + i\left({}^*f^{123} -{}^* f^{156} +{}^* f^{246} -{}^* f^{345}\right)\right\} \tag{C.5}
$$

$$f^{x\overline{y}\overline{z}} = \frac{1}{8}\left\{f^{123} - f^{156} - f^{246} + f^{345} + i\left({}^*f^{123} -^* f^{156} -^* f^{246} +^* f^{345}\right)\right\} \qquad \text{(C.6)}$$

$$f^{xy\overline{z}} = \frac{1}{8}\left\{f^{123} + f^{156} - f^{246} - f^{345} + i\left({}^*f^{123} +^* f^{156} -^* f^{246} -^* f^{345}\right)\right\} \qquad \text{(C.7)}$$

$$f^{x\overline{y}k} = \frac{1}{8}\left\{f^{123} + f^{156} + f^{246} + f^{345} + i\left({}^*f^{123} +^* f^{156} +^* f^{246} +^* f^{345}\right)\right\} \qquad \text{(C.8)}$$

while

$$f^{x\overline{x}y} = f^{x\overline{x}z} = f^{y\overline{y}x} = f^{y\overline{y}z} = f^{z\overline{z}x} = f^{z\overline{z}y} = 0 \qquad \text{(C.9)}$$

and therefore, the twenty entries of $f_1^{\Lambda\Sigma\Delta}$ are reduced to eight.

In this holomorphic basis the gravitino mass eigenvalues assume the rather simple form:

$$m_1 \equiv |\mu_1 + i\mu_1^{'}| = \frac{1}{6|L^2|}|f^{x\overline{y}\overline{z}}| \qquad \text{(C.10)}$$

$$m_2 \equiv |\mu_2 + i\mu_2^{'}| = \frac{1}{6|L^2|}|f^{x\overline{y}z}| \qquad \text{(C.11)}$$

$$m_3 \equiv |\mu_3 + i\mu_3^{'}| = \frac{1}{6|L^2|}|f^{xy\overline{z}}| \qquad \text{(C.12)}$$

$$m_4 \equiv |\mu_4 + i\mu_4^{'}| = \frac{1}{6|L^2|}|f^{xyz}|\,. \qquad \text{(C.13)}$$

We note that the three axions $B^{\Lambda\Sigma} = \{B_{14},\, B_{25},\, B_{36}\} \equiv \{-2iB_{x\overline{x}},\, -2iB_{y\overline{y}},\, -2iB_{z\overline{z}}\}$ are inert under T_{12}-gauge transformations, since we have 15 axions but only 12 bulk vectors.

When we consider the truncation to the $N = 3$ theory we expect that only 9 complex scalar fields become massless moduli parametrizing $\mathrm{SU}(3,3)/\mathrm{SU}(3) \times \mathrm{SU}(3) \times \mathrm{U}(1)$. Moreover, it is easy to see that if we set e.g. $\mu_1 = \mu_2 = \mu_3 = 0$ ($\mu_1^{'} = \mu_2^{'} = \mu_3^{'} = 0$) which implies $f^{345} = f^{156} = -f^{123} = -f^{246}$ (${}^*f^{345} =^* f^{156} = -^*f^{123} = -^*f^{246}$) in the $N = 3$ theory, we get that also the 6 fields $B^{12} - B^{45}$, $B^{13} - B^{46}$, $B^{24} - B^{15}$, $B^{34} - B^{16}$, $B^{23} - B^{56}$, $B^{35} - B^{26}$ are inert under gauge transformations.

In holomorphic coordinates, the translational gauging implies that the differential of the axionic fields become covariant and they are given by:

$$\nabla_{(g)}B^{ij} \equiv dB^{ij} + (\mathrm{Re}\, f^{ijk1})A_{k1} + (\mathrm{Re}\, f^{ij\overline{k}1})A_{\overline{k}1} + (\mathrm{Im}\, f^{ijk2})A_{k2} + (\mathrm{Im}\, f^{ij\overline{k}2})A_{\overline{k}2} \quad \text{(C.14)}$$

$$\nabla_{(g)}B^{i\overline{\jmath}} \equiv dB^{i\overline{\jmath}} + (\mathrm{Re}\, f^{i\overline{\jmath}k1})A_{k1} + (\mathrm{Re}\, f^{i\overline{\jmath}\overline{k}1})A_{\overline{k}1} + (\mathrm{Im}\, f^{i\overline{\jmath}k2})A_{k2} + (\mathrm{Im}\, f^{i\overline{\jmath}\overline{k}2})A_{\overline{k}2}\,. \quad \text{(C.15)}$$

Since in the $N = 4 \longrightarrow N = 3$ truncation the only surviving massless moduli fields are $B_{i\overline{\jmath}} + ig_{i\overline{\jmath}}$, then the 3+3 axions $\{B_{ij},\, B_{\overline{\imath}\overline{\jmath}}\}$ give mass to 6 vectors, while $\delta B_{i\overline{\jmath}}$ must be zero. We see from equation (C.15) we see that we must put to zero the components

$$f^{i\overline{\jmath}k} = f^{i\overline{\jmath}\overline{k}} = f^{ij\overline{k}} = 0 \qquad \text{(C.16)}$$

while

$$f^{ijk} \equiv f\epsilon^{ijk} \neq 0\,. \qquad \text{(C.17)}$$

Looking at the equations (C.10) we see that these relations are exactly the same which set $\mu_1 + i\mu_1^{'} = \mu_2 + i\mu_2^{'} = \mu_3 + i\mu_3^{'} = 0$ and $\mu_4 + i\mu_4^{'} \neq 0$, confirming that the chosen complex structure corresponds to the $N = 3$ theory. Note that the corresponding $g^{i\overline{\jmath}}$ fields partners of $B^{i\overline{\jmath}}$ in the chosen complex structure parametrize the coset $\mathrm{O}(1,1) \times \mathrm{SL}(3,\mathbb{C})/\mathrm{SU}(3)$. Actually the freezing of the holomorphic g^{ij} gives the following relations among the components in the real basis of $g^{\Lambda\Sigma}$:

$$g^{14} = g^{25} = g^{36} = 0 \tag{C.18}$$

$$g^{11} - g^{44} = 0\,, \qquad g^{22} - g^{55} = 0\,, \qquad g^{33} - g^{66} = 0 \tag{C.19}$$

$$g^{12} - g^{45} = 0\,, \qquad g^{13} - g^{46} = 0\,, \qquad g^{23} - g^{56} = 0 \tag{C.20}$$

$$g^{15} + g^{24} = 0\,, \qquad g^{16} + g^{34} = 0\,, \qquad g^{26} + g^{35} = 0\,. \tag{C.21}$$

The freezing of the axions B^{ij} in the holomorphic basis give the analogous equations:

$$B^{12} - B^{45} = 0\,, \qquad B^{13} - B^{46} = 0\,, \qquad B^{23} - B^{56} = 0 \tag{C.22}$$

$$B^{15} + B^{42} = 0\,, \qquad B^{16} + B^{43} = 0\,, \qquad B^{26} + B^{53} = 0 \tag{C.23}$$

$$B^{14} = B^{25} = B^{36} = 0\,. \tag{C.24}$$

The massless $g^{i\overline{\jmath}}$ and $B^{i\overline{\jmath}}$ are instead given by the following combinations:

$$g^{x\overline{x}} = \frac{1}{2}(g^{11} + g^{44})\,, \qquad g^{y\overline{y}} = \frac{1}{2}(g^{22} + g^{55})\,, \qquad g^{z\overline{z}} = \frac{1}{2}(g^{33} + g^{66}) \tag{C.25}$$

$$B^{x\overline{x}} = \frac{i}{2}B^{14}\,, \qquad B^{y\overline{y}} = \frac{i}{2}B^{25}\,, \qquad B^{z\overline{z}} = \frac{i}{2}B^{36} \tag{C.26}$$

$$g^{x\overline{y}} = \frac{1}{2}(g^{12} + ig^{15})\,, \qquad g^{x\overline{z}} = \frac{1}{2}(g^{13} + ig^{16})\,, \qquad g^{y\overline{z}} = \frac{1}{2}(g^{23} + ig^{26}) \tag{C.27}$$

$$B^{x\overline{y}} = \frac{1}{2}(B^{12} + iB^{15}), \qquad B^{x\overline{z}} = \frac{1}{2}(B^{13} + iB^{16}), \qquad B^{y\overline{z}} = \frac{1}{2}(B^{23} + iB^{26}) \tag{C.28}$$

$$B^{xx} = B^{yy} = B^{zz} = 0 \tag{C.29}$$

Let us now consider the reduction $N = 4 \longrightarrow N = 2$ for which the relevant moduli space is $\mathrm{SU}(2,2)/\left(\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{U}(1)\right) \otimes \mathrm{SU}(1,1)/\mathrm{U}(1)$. Setting $\mu_2 + i\mu_2^{'} = \mu_3 + i\mu_3^{'} = 0$ we find:

$$f^{x\overline{y}z} = f^{xy\overline{z}} = 0 \tag{C.30}$$

which, in real components implies:

$$f^{123} + f^{156} = 0\,; \qquad f^{246} + f^{345} = 0 \tag{C.31}$$

and analogous equations for their Hodge dual. This implies that in the $N = 2$ phase two more axions are gauge inert namely $B^{23} + B^{56} = 2B^{23}$ and $B^{26} + B^{35} = 2B^{26}$ or, in holomorphic components, $B^{y\overline{z}}$. The remaining fields are $g^{14}, g^{25}, g^{36}, g^{23}, g^{26}$ and $B^{14}, B^{25}, B^{36}, B^{23}, B^{26}$, the last ones parametrize the coset $\mathrm{SO}(1,1) \times \mathrm{SO}(2,2)/\mathrm{SO}(2) \times \mathrm{SO}(2)$.

If we now consider the truncation $N = 4 \longrightarrow N = 1$ the relevant coset manifold is $(\mathrm{SU}(1,1)/\mathrm{U}(1))^3$ which contains 3 complex moduli. To obtain the corresponding complex structure, it is sufficient to freeze $g^{i\overline{\jmath}}$, $B^{i\overline{\jmath}}$ with $i \neq j$. In particular the $\mathrm{SU}(1,1)^3$ can be decomposed into $\mathrm{O}(1,1)^3 \otimes^s T^3$ where the three $\mathrm{O}(1,1)$ and the three translations T^3 are parametrized by $g^{x\overline{x}}$, $g^{y\overline{y}}$, $g^{z\overline{z}}$ and $B^{x\overline{x}}$, $B^{y\overline{y}}$, $B^{z\overline{z}}$ respectively.

These axions are massless because of equation (C.9) (Note that the further truncation $N = 1 \longrightarrow N = 0$ does not alter the coset manifold $\mathrm{SU}(1,1)^3$ since we have no loss of massless fields in this process). In this case we may easily compute the moduli dependence of the gravitino masses. Indeed, $\mathrm{O}(1,1)^3$, using equations (C.19), (C.25), will have as coset representative the matrix

$$E^I_\Lambda = \begin{pmatrix} e^{\varphi_1} & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{\varphi_2} & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{\varphi_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{\varphi_1} & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{\varphi_2} & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{\varphi_3} \end{pmatrix} \tag{C.32}$$

where we have set $g_{11} = e^{2\varphi_1}$, $g_{22} = e^{2\varphi_2}$, $g_{33} = e^{2\varphi_3}$, the exponentials representing the radii of the manifold $T^2_{(14)} \times T^2_{(25)} \times T^2_{(36)}$.

We see that in the gravitino mass formula the vielbein E^I_Λ reduces to the diagonal components of the matrix (C.32) A straightforward computation then gives:

$$S_{AB}\overline{S}^{AB} = \frac{2}{(48)^2} e^{(2\varphi_1+2\varphi_2+2\varphi_3)} \begin{pmatrix} m_1^2 & 0 & 0 & 0 \\ 0 & m_2^2 & 0 & 0 \\ 0 & 0 & m_3^2 & 0 \\ 0 & 0 & 0 & m_4^2 \end{pmatrix}. \tag{C.33}$$

We note that in the present formulation where we have used a contravariant $B^{\Lambda\Sigma}$ as basic charged fields, the gravitino mass depends on the T^6 volume. However if we made use of the dual 4-form $C_{\Lambda\Sigma\Gamma\Delta}$, as it comes from type-IIB string theory, then the charge coupling would be given in terms of ${}^*f^\alpha_{\Lambda\Sigma\Gamma}$ and the gravitino mass matrix would be trilinear in E^Λ_I instead of E^I_Λ. Therefore all our results can be translated in the new one by replacing $R_i \to R_i^{-1}$.

D. Conventions

We realize the isomorphism between the two fold antisymmetric representation of SU(4) and the fundamental of SO(6) using the 4×4 Γ-matrix $(\Gamma^I)_{AB} = -(\Gamma^I)_{BA}$.

We have used the following representation

$$\Gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \qquad \Gamma^4 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}$$
$$\Gamma^2 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \qquad \Gamma^5 = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \\ -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix} \tag{D.1}$$

$$\Gamma^3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \qquad \Gamma^6 = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix}.$$

Note that their anticommutator is $\{\Gamma_I, \overline{\Gamma}_J\} = -\delta_{IJ}$, where the complex conjugation acts as

$$(\Gamma^I)^{AB} = \left(\overline{\Gamma}^I\right)_{AB} = \frac{1}{2}\epsilon^{ABCD}(\Gamma^I)_{CD}. \tag{D.2}$$

We define

$$(\Gamma^{IJ})_A{}^B = \frac{1}{2}\left[\left(\Gamma^{[I}\right)_{AC}\left(\Gamma^{J]}\right)^{CB}\right] \tag{D.3}$$

$$(\Gamma^{IJK})_{AB} = \frac{1}{3!}\left[(\Gamma^I)_{AC}(\Gamma^J)^{CD}(\Gamma^K)_{DB} + \text{perm.}\right]. \tag{D.4}$$

Here the matrices $(\Gamma^{IJK})_{AB}$ are symmetric and satisfy the relation

$$\left(\Gamma^{IJK}\right)_{AB} = \frac{i}{6}\varepsilon^{IJKLMN}(\Gamma_{LMN})_{AB}. \tag{D.5}$$

In this representation, the following matrices are diagonal:

$$\Gamma^{123} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \Gamma^{156} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\Gamma^{246} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \Gamma^{345} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{D.6}$$

as well as the matrices Γ^{456}, Γ^{234}, Γ^{135} and Γ^{126} related with them through the relation (D.5).

We define for a generic tensor

$$T_{\dots I\,J\dots} = \dots\frac{1}{2}(\Gamma_I)^{AB}\frac{1}{2}(\Gamma_J)^{CD}\dots T_{\dots[AB]\,[CD]\dots}$$

$$T_{\dots[AB]\,[CD]\dots} = \dots\frac{1}{2}(\Gamma^I)_{AB}\frac{1}{2}(\Gamma^J)_{CD}\dots T_{\dots I\,J\dots} \tag{D.7}$$

so that

$$E^I_\Lambda(E^{-1})^\Sigma_I = E^{AB}_\Lambda(E^{-1})^\Sigma_{AB} = \delta^\Sigma_\Lambda \tag{D.8}$$

$$E^I_\Lambda(E^{-1})^\Lambda_J = \delta^J_I \iff E^{AB}_\Lambda(E^{-1})^\Lambda_{CD} = \delta^{AB}_{CD} \tag{D.9}$$

In particular we need to convert the $\mathrm{SO}(6)_1$ indices of ω_1^{IJ} into $\mathrm{SU}(4)$ R-symmetry indices as they appear in the covariant derivative on spinors. For this purpose we apply the previous definition (D.7) to the connection ω_1^{IJ} defining

$$\omega^{AB}{}_{CD} \equiv \frac{1}{4}(\Gamma_I)^{AB}(\Gamma_J)_{CD}\,\omega_1^{IJ} \tag{D.10}$$

then we introduce the connection $Q^A{}_B$ defined as

$$\omega^{AB}{}_{CD} \equiv \delta^{[A}_{[C} Q^{B]}{}_{D]} \tag{D.11}$$

and thus

$$Q^A{}_D = -\frac{1}{2}\omega^{AB}{}_{BD} = \frac{1}{8}(\Gamma_{IJ})^A{}_D\, \omega_1^{IJ} \tag{D.12}$$

One can easily realize that given the definition of the $\mathrm{SO}(6)_1$ curvature as

$$R_1^{IJ} \equiv d\omega_1^{IJ} + \omega_1^I{}_K \wedge \omega_1^{KJ} \tag{D.13}$$

one finds for consistence that

$$R^A{}_B \equiv dQ^A{}_B - 2Q^A{}_C \wedge Q^C{}_B = \frac{1}{8}(\Gamma_{IJ})^A{}_B\, R_1^{IJ}\,. \tag{D.14}$$

As a consequence, the covariant derivative acting on spinors turns out to be

$$D\theta_A = d\theta_A - 2Q^A{}_B\theta_B = d\theta_A - \frac{1}{4}(\Gamma_{IJ})^A{}_B\, \omega_1^{IJ}\theta_B\,. \tag{D.15}$$

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